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Note

Improved bounds on acyclic edge colouring

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Abstract

We prove that the acyclic chromatic index $a'(G) \leqslant 6\Delta$ for all graphs with girth at least 9. We extend the same method to obtain a bound of 4.52Δ with the girth requirement $g \geqslant 220$. We also obtain a relationship between g and $a'(G)$.

Keywords: Acyclic graph edge colouring; Graph; Edge chromatic index

1. Introduction

We consider only simple undirected graphs. Throughout the paper, we use $\Delta = \Delta(G)$ to denote the maximum degree of a graph G and $g = g(G)$ to denote the girth (length of a shortest cycle) of G. A colouring of the edges of a graph is *proper* if no pair of incident edges receive the same colour. A proper colouring of the edges of a graph G is called *acyclic* if there is no bichromatic (two-coloured) cycle in G. In other words, the subgraph induced by the union of any two colour classes is a forest. The minimum number of colours required for any acyclic edge colouring is called the *acyclic chromatic index* of G and is denoted by $a'(G)$. This notion was introduced by Grunbaum in [6].

It is obvious that any proper edge colouring of G requires at least Δ colours, and Vizing [9] showed that there exists a proper edge colouring with $\Delta + 1$ colours. Using probabilistic arguments, Alon et al. [1] obtained a bound of 64Δ on $a'(G)$ which was later improved to 16Δ (presently best known upper bound) by Molloy and Reed in [7]. In [8, Chapter 19, p. 226], a bound of 9Δ is claimed for $a'(G)$ but the proof is incorrect and is not easily rectifiable (see the Appendix A for details).

In this work we obtain a bound of 4.52Δ for all graphs with $g(G) \geqslant 220$. We can relax the girth requirement to 9 if we are willing to use 6Δ colours. It might be possible to remove the girth requirement with a more sophisticated analysis.

Theorem 1. *If* $g(G) \geqslant 9$, *then* $a'(G) \leqslant 6\Delta$.

Theorem 2. *If* $g(G) \geqslant 220$, *then* $a'(G) \leqslant 4.52\Delta$.

Our proofs are based on probabilistic arguments. We make use of the same random experiment used in [1,7].

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The main new idea of the proof is to allow limited improperness with acyclicity in the first phase of colouring and take care of properness in the second phase. The reason we need to allow improperness is that, without this, one can show (as explained in the Appendix A) that one will have to use more than $4e\Delta$ colours. Both theorems are proved using essentially the same arguments.

It is conjectured that $a'(G)\leqslant\Delta+2$ in [3] and in the same work this conjecture is proved for graphs with $g=\Omega(\Delta\log\Delta)$ and also that $a'(G)\leqslant 2\Delta+2$ for graphs with $g=\Omega(\log\Delta)$. However, when using probabilistic arguments, short cycles are the major obstacles since they have a "high" probability of becoming bichromatic compared to long cycles. Hence, bounding $a'(G)$ for graphs with short cycles seems fairly difficult. The following theorem tries to capture this phenomenon in a formal way. As a corollary, we notice that $a'(G)\leqslant\Delta+\mathrm{o}(\Delta)$ for all graphs G with $g=\omega(\log\Delta)$. All logarithms are to the base e.

Theorem 3. *There are absolute constants $c_1, c_2>0$ such that, for any G with $g\geqslant c_1\log\Delta$ we have*

$$a'(G)\leqslant\Delta+1+\left\lceil c_2\left(\frac{\Delta\log\Delta}{g}\right)\right\rceil.$$

We make use of Lovász Local Lemma [5,8,2] stated below.

Lemma 4 (*The Lovász Local Lemma*). *Let $\mathscr{A}=\{A_1,\ldots,A_n\}$ be events in a probability space Ω such that each event A_i is mutually independent of $\mathscr{A}-(\{A_i\}\cup\mathscr{D}_i)$, for some $\mathscr{D}_i\subseteq\mathscr{A}$. Also suppose that there exist $x_1,\ldots,x_n\in(0,1)$ such that*

$$\Pr(A_i)\leqslant x_i\prod_{A_j\in\mathscr{D}_i}(1-x_j),\quad 1\leqslant i\leqslant n.$$

Then $\Pr(\overline{A_1}\wedge\cdots\wedge\overline{A_n})>0$.

In Section 2, we prove Theorems 1 and 2. In Section 3, we prove Theorem 3. Section 4 concludes with some remarks.

2. Acyclic colouring for graphs with constant girth

We have not optimised the constants in the application of Lovász Local Lemma. With a more careful application of local lemma it might be possible to bring down the bound a bit further.

Proof of Theorems 1 and 2. It is known that, if $\Delta\leqslant 3$, then $a'(G)\leqslant\Delta+2$ [4]. Hence we may assume that $\Delta\geqslant 4$ in our arguments. Our proof consists of two stages. In the first stage, we show, by probabilistic arguments, the existence of a colouring $\mathscr{C}$, using a set C of $c\Delta$ colours (where $c>1$ is a constant to be fixed later), such that $\mathscr{C}$ satisfies the following properties for some positive integer $\eta\leqslant 4$:

(i) every vertex has at most η incident edges of any single colour,
(ii) there are no properly two-coloured cycles, and
(iii) there are no monochromatically coloured cycles.

Note that in $\mathscr{C}$ each *colour class* (set of edges receiving the same colour) is a forest of maximum degree at most η. In the second stage we split each colour class into η parts by recolouring the edges of each colour c_i with the colours $c_i^1,\ldots,c_i^\eta$ to get a colouring $\mathscr{C}'$. We claim that $\mathscr{C}'$ is proper and acyclic. Since every forest of maximum degree at most d is properly edge colourable using d colours, it is easy to see that properness holds. Any bichromatic cycle in the colouring $\mathscr{C}'$ should either come from an existing two-coloured cycle in $\mathscr{C}$, or from a monochromatic even length cycle in $\mathscr{C}$ being split into two. Both of these possibilities are forbidden by properties (ii) and (iii), respectively. It follows that the colouring $\mathscr{C}'$ is proper, acyclic, and uses at most $c\eta\Delta$ colours.

To complete the proof, it is now sufficient to show that such a colouring $\mathscr{C}$ described above exists with positive probability. We do this using Lovász Local Lemma. For this, we do the following random experiment. Each edge chooses a colour uniformly and independently at random, from the set C. For the resulting random colouring to satisfy (i)–(iii) above, define the following three types of *unfavourable* events. As explained below, in the absence of these events, the colouring obtained satisfies the above properties:

(1) For a set of $\eta+1$ edges $\{e_1,\ldots,e_{\eta+1}\}$ incident on a vertex u, let $E_{e_1,\ldots,e_{\eta+1}}$ be the event that all of them receive the same colour. We call this an event of type I.
(2) Let $E_{C,2k}$ denote the event that an even cycle C of length $2k$ is properly coloured with two colours. We call this an event of type II.
(3) Let $E_{C,\ell}$ denote the event that a cycle C of length ℓ is coloured monochromatically. We call this an event of type III.

Suppose $\mathscr{C}$ be such that none of the above events holds. We claim that properties (i)–(iii) above are satisfied. It is easy to see that the absence of events of type I implies that (i) holds. Similarly, absence of type II and III events, respectively, implies (ii) and (iii).

In order to apply the local lemma, we need estimates for the probabilities of each event, and also for the number of other events of each type which can possibly influence any given event. For the above random experiment, an event $\mathscr{E}$ is mutually independent of a set $\mathscr{B}$ of other events if the set of edges on which $\mathscr{E}$ depends is disjoint from the set of edges on which the events in $\mathscr{B}$ depend. Hence, we calculate the number of events of each type that depend on a given edge, and multiply by the number of edges to get an upper bound on the number of events influencing $\mathscr{E}$. The following two lemmas present the estimated bounds. The proof of Lemma 5 is straightforward. Lemma 6 is also not difficult and uses standard arguments (see [1] for details).

Lemma 5. *The probabilities of events are as follows*:

1. *For each event* $E_{e_1,\ldots,e_{\eta+1}}$ *of type* I, $\Pr(E_{e_1,\ldots,e_{\eta+1}}) = 1/|C|^{\eta}$.
2. *For each event* $E_{C,2k}$ *of type* II, *where length of* C *is* $2k$, $\Pr(E_{C,2k}) \leqslant 1/|C|^{2k-2}$.
3. *For each event* $E_{C,\ell}$ *of type* III, *where* C *is of length* ℓ, $\Pr(E_{C,\ell}) = 1/|C|^{\ell-1}$.

Lemma 6. *The following is true for any given edge* e:

1. *Less than* $2\Delta^{\eta}/\eta!$ *events of type* I *depend on* e.
2. *Less than* Δ^{2k-2} *events of type* II *depend on* e.
3. *Less than* $\Delta^{\ell-2}$ *events of type* III *depend on* e.

In order to apply Lovász Local Lemma, let $x_0 = 1/(\alpha\Delta)^{\eta}$, $x_k = 1/(\beta\Delta)^{2k-2}$, and $y_\ell = 1/(\gamma\Delta)^{\ell-1}$ be the values associated with events of types I, II, and III respectively, where $\alpha,\beta,\gamma > 1$ are constants to be determined later. Recall that we use g to denote girth. We conclude that with positive probability none of the above events occur, provided that $\forall k \geqslant \lceil g/2\rceil, \ell \geqslant g$,

$$\frac{1}{(c\Delta)^{\eta}} \leqslant x_0(1-x_0)^{(\eta+1)2\Delta^{\eta}/\eta!} \prod_{\theta\geqslant\lceil g/2\rceil} (1-x_\theta)^{(\eta+1)\Delta^{2\theta-2}} \prod_{\lambda\geqslant g} (1-y_\lambda)^{(\eta+1)\Delta^{\lambda-2}},$$

$$\frac{1}{(c\Delta)^{2k-2}} \leqslant x_k(1-x_0)^{2k(2\Delta^{\eta}/\eta!)} \prod_{\theta\geqslant\lceil g/2\rceil} (1-x_\theta)^{2k\Delta^{2\theta-2}} \prod_{\lambda\geqslant g} (1-y_\lambda)^{2k\Delta^{\lambda-2}},$$

$$\frac{1}{(c\Delta)^{\ell-1}} \leqslant y_\ell(1-x_0)^{\ell(2\Delta^{\eta}/\eta!)} \prod_{\theta\geqslant\lceil g/2\rceil} (1-x_\theta)^{\ell\Delta^{2\theta-2}} \prod_{\lambda\geqslant g} (1-y_\lambda)^{\ell\Delta^{\lambda-2}}.$$

Let $f(z) = (1-1/z)^z$. It is well known that $(1-1/z)^z \uparrow 1/e$. Defining

$$\Lambda = \min\left\{ f(x_0^{-1}),\ \min_{\theta\geqslant\lceil g/2\rceil} f(x_\theta^{-1}),\ \min_{\lambda\geqslant g} f(y_\lambda^{-1}) \right\},$$

it follows that

$$(1-x_0)^{2\Delta^\eta/\eta!} = \left(1-\frac{1}{(\alpha\Delta)^\eta}\right)^{2\Delta^\eta/\eta!} = \left(\left(1-\frac{1}{(\alpha\Delta)^\eta}\right)^{(\alpha\Delta)^\eta}\right)^{2/(\eta!\alpha^\eta)} \geqslant \Lambda^{2/(\eta!\alpha^\eta)}.$$

Similarly,

$$\prod_{\theta\geqslant\lceil g/2\rceil}(1-x_\theta)^{\Delta^{2\theta-2}} = \prod_{\theta\geqslant\lceil g/2\rceil}\left(1-\frac{1}{(\beta\Delta)^{2\theta-2}}\right)^{\Delta^{2\theta-2}} \geqslant \prod_{\theta\geqslant\lceil g/2\rceil}\Lambda^{\beta^{-(2\theta-2)}} \geqslant \Lambda^{S_1},$$

where

$$S_1 = \sum_{\theta\geqslant\lceil g/2\rceil}\frac{1}{\beta^{2\theta-2}} \leqslant \frac{1}{(\beta^2-1)\beta^{2\lceil g/2\rceil-4}} \quad \text{and}$$

$$\prod_{\lambda\geqslant g}(1-y_\lambda)^{\Delta^{\lambda-2}} = \prod_{\lambda\geqslant g}\left(1-\frac{1}{(\gamma\Delta)^{\lambda-1}}\right)^{\Delta^{\lambda-2}} \geqslant \prod_{\lambda\geqslant g}\Lambda^{\gamma^{-(\lambda-1)}/\Delta} \geqslant \Lambda^{S_2},$$

where

$$S_2 = \sum\nolimits_{\lambda\geqslant g}\frac{1}{\Delta\gamma^{\lambda-1}} \leqslant \frac{1}{\Delta\gamma^{g-2}(\gamma-1)}.$$

Thus, taking roots on both sides and simplifying, the three inequalities required by local lemma are satisfied $\forall k\geqslant\lceil g/2\rceil$, $\ell\geqslant g$, provided that

$$\frac{1}{c}\leqslant\frac{1}{\alpha}\Lambda^{((\eta+1)/\eta)\Upsilon}, \quad \frac{1}{c}\leqslant\frac{1}{\beta}\Lambda^{(2k/(2k-2))\Upsilon} \quad \text{and} \quad \frac{1}{c}\leqslant\frac{1}{\gamma}\Lambda^{(\ell/(\ell-1))\Upsilon}, \tag{1}$$

where

$$\Upsilon = \frac{2}{\eta!\alpha^\eta} + \frac{1}{(\beta^2-1)\beta^{2\lceil g/2\rceil-4}} + \frac{1}{\Delta\gamma^{g-2}(\gamma-1)}.$$

Now we have to set specific values of α, β, γ, and η. Firstly, we set $\eta=2$ and $\alpha=\beta=\gamma=2$. Using $g\geqslant 9$ and $\Delta\geqslant 4$, we have $\Lambda\geqslant(1-\frac{1}{64})^{64}\geqslant 0.3649$. It can easily be verified that the above inequalities (1) are satisfied by setting $c=2.951$. It follows that $a'(G)\leqslant 5.91<6\Delta$ for all graphs G with girth $g\geqslant 9$. This proves Theorem 1.

Secondly, we set $\eta=4$, $\alpha=1.02$, $\beta=1.04$, and $\gamma=1.04$. Using $g\geqslant 220$ and $\Delta\geqslant 4$, we have $\Lambda\geqslant(1-\frac{1}{256})^{256}\geqslant 0.3671$. It follows that by setting $c=1.13$, $a'(G)\leqslant 4\times 1.13\Delta=4.52\Delta$ when girth $g\geqslant 220$. Hence Theorem 2. □

Further improvements on $a'(G)$, which can be obtained (with this experiment) by strengthening the girth requirement, are only marginal as long as we focus on constant lower bounds on girth.

3. A general relation between $g(G)$ and $a'(G)$

An even cycle is called *half-monochromatic* with respect to a colouring if one of its *halves* (a set of alternate edges) is monochromatic. Notice that this definition includes bichromatic cycles also.

Proof of Theorem 3. For the sake of simplicity in the analysis, we write g in the form $c_1\Delta^\varepsilon\log\Delta$, where $\varepsilon\geqslant 0$ and where c_1 is mentioned in Theorem 3. We can assume $w.l.o.g.$ that $\varepsilon\leqslant 1$, because when ε exceeds 1, by choosing a large value of c_1, $a'(G)\leqslant\Delta+2$ as shown in [3]. As before, we assume $\Delta\geqslant 4$.

The proof consists of an initial deterministic phase followed by a random phase. We begin by obtaining a proper edge colouring of G using $\Delta+1$ colours applying Vizing's method. We then randomly recolour some of the edges with a new set of $o(\Delta)$ colours, and show that with positive probability, the colouring obtained is proper and acyclic. This random experiment is a slight modification of the one used in [3].

The random colouring is obtained as follows:

(1) Obtain a proper colouring $\mathscr{C} : E \to S_1 = \{1, \ldots, \Delta + 1\}$.
(2) In the second phase we do the following:
- Activate each edge with independent probability $p = 1/\Delta^{\varepsilon}$.
- Each activated edge chooses a new colour uniformly at random and independently, from the set $S_2 = \{1', \ldots, (a\Delta^{1-\varepsilon})'\}$, where $a > 1$ is a constant to be determined later.

Denote the resulting random colouring by $\mathscr{C}'$. With respect to $\mathscr{C}'$, we define the following *unfavourable* events:

(1) For a pair of incident edges e and f, let $E_{e,f}$ denote the event that they are both recoloured with the same new colour. We call this an event of type I.
(2) Let $E_{C,2k}$ denote the event that a bichromatic cycle C of length $2k$ in $\mathscr{C}$ is undisturbed in the recolouring process. Call this a type II event.
(3) Let $E_{C,2\ell}$ denote the event that a half-monochromatic cycle C of length 2ℓ in $\mathscr{C}$ becomes bichromatic by retaining the same colour on a half and receiving a common new colour on the other half, a type III event.
(4) Let $E_{C,2m}$ denote the type IV event where an even length cycle C of length $2m$ becomes properly bichromatic with two of the new colours.

We claim that the absence of types I–IV events implies that the colouring $\mathscr{C}'$ is proper and is also acyclic. Since $\mathscr{C}$ is proper, the absence of events of type I ensures that $\mathscr{C}'$ is also proper. The absence of events of types II, III, and IV ensures, respectively, (i) the absence of bichromatic cycles using both colours from S_1, (ii) one colour from each of S_1 and S_2, and (iii) both colours from S_2. It is therefore sufficient to show the absence of the above four types of events which we do by using Lovász Local Lemma.

To apply the local lemma we need estimates for the probabilities of each event, and for the number of events of each type possibly influencing a given event. As before, we calculate the number of events of each type that depend on a single edge and multiply by the number of edges in any event to get an upper bound on the total dependency. The following two lemmas present the estimated bounds.

Lemma 7. *The probabilities of events are as follows: for each*

1. *event* $E_{f,g}$ *of type* I, $\Pr(E_{f,g}) = p^2/(a\Delta^{1-\varepsilon}) = 1/(a\Delta^{1+\varepsilon})$.
2. *event* $E_{C,2k}$ *of type* II, $\Pr(E_{C,2k}) = (1-p)^{2k} \leqslant \mathrm{e}^{-2k/\Delta^{\varepsilon}}$.
3. *event* $E_{C,2\ell}$ *of type* III, $\Pr(E_{C,2\ell}) \leqslant 2p^{\ell}(1-p)^{\ell}/(a\Delta^{1-\varepsilon})^{\ell-1} < 2a\Delta^{1-\varepsilon}/(a\Delta)^{\ell}$.
4. *event* $E_{C,2m}$ *of type* IV, $\Pr(E_{C,2m}) = p^{2m}\binom{a\Delta^{1-\varepsilon}}{2}(2/(a\Delta^{1-\varepsilon})^{2m}) < (a\Delta^{1-\varepsilon})^2/(a\Delta)^{2m}$.

Lemma 8. *The following is true for any given edge* e:

1. *Less than* 2Δ *events of type* I *depend on* e.
2. *Less than* Δ *events of type* II *depend on* e.
3. *Less than* $2\Delta^{\ell-1}$ *events of type* III *depend on* e, *for each* $\ell \geqslant 2$.
4. *Less than* Δ^{2m-2} *events of type* IV *depend on* e, *for each* $m \geqslant 2$.

To apply Lovász Local Lemma, let $x_0 = 1/(\alpha\Delta^{1+\varepsilon})$, $x_1 = 1/(\beta\Delta^{1+2\varepsilon})$, $y_\ell = (2a\Delta^{1-\varepsilon})/(\gamma\Delta)^{\ell}$, and $z_m = (a\Delta^{1-\varepsilon})^2/((\delta\Delta)^{2m})$ be the values associated with events of types I–IV, where lengths of cycles III and IV are 2ℓ and $2m$, respectively. Here $\alpha, \beta, \gamma, \delta > 1$ are real values to be determined later. We conclude that with positive probability none of the above events occurs, provided that $\forall k, \ell, m \geqslant \lceil g/2 \rceil$,

$$\frac{1}{a\Delta^{1+\varepsilon}} \leqslant x_0(1-x_0)^{4\Delta}(1-x_1)^{2\Delta} \prod_{\theta \geqslant \lceil g/2 \rceil} (1-y_\theta)^{4\Delta^{\theta-1}} \prod_{\lambda \geqslant \lceil g/2 \rceil} (1-z_\lambda)^{2\Delta^{2\lambda-2}},$$

$$\mathrm{e}^{-2k/\Delta^{\varepsilon}} \leqslant x_1(1-x_0)^{4k\Delta}(1-x_1)^{2k\Delta} \prod_{\theta \geqslant \lceil g/2 \rceil} (1-y_\theta)^{4k\Delta^{\theta-1}} \prod_{\lambda \geqslant \lceil g/2 \rceil} (1-z_\lambda)^{2k\Delta^{2\lambda-2}},$$

$$\frac{2a\Delta^{1-\varepsilon}}{(a\Delta)^{\ell}} \leqslant y_\ell(1-x_0)^{4\ell\Delta}(1-x_1)^{2\ell\Delta} \prod_{\theta \geqslant \lceil g/2 \rceil} (1-y_\theta)^{4\ell\Delta^{\theta-1}} \prod_{\lambda \geqslant \lceil g/2 \rceil} (1-z_\lambda)^{2\ell\Delta^{2\lambda-2}},$$

$$\frac{(a\Delta^{1-\varepsilon})^2}{(a\Delta)^{2m}} \leqslant z_m(1-x_0)^{4m\Delta}(1-x_1)^{2m\Delta} \prod_{\theta \geqslant \lceil g/2 \rceil} (1-y_\theta)^{4m\Delta^{\theta-1}} \prod_{\lambda \geqslant \lceil g/2 \rceil} (1-z_\lambda)^{2m\Delta^{2\lambda-2}}.$$

Setting $\alpha=\beta=\gamma=\delta=1000$ and $a=4000$ and using the fact that $(1-1/z)^z \geqslant \frac{1}{4}$ $\forall z \geqslant 2$ we have

$$(1-x_0)^{2\Delta} \geqslant \left(\frac{1}{4}\right)^{2\Delta x_0} = \left(\frac{1}{4}\right)^{2/(\alpha\Delta^{\varepsilon})}, \quad (1-x_1)^{\Delta} \geqslant \left(\frac{1}{4}\right)^{\Delta x_1} = \left(\frac{1}{4}\right)^{1/(\beta\Delta^{2\varepsilon})},$$

$$\prod_{\theta \geqslant \lceil g/2 \rceil} (1-y_\theta)^{2\Delta^{\theta-1}} \geqslant \left(\frac{1}{4}\right)^{\mathscr{S}_1} \quad \text{and} \quad \prod_{\lambda \geqslant \lceil g/2 \rceil} (1-z_\lambda)^{\Delta^{2\lambda-2}} \geqslant \left(\frac{1}{4}\right)^{\mathscr{S}_2},$$

where

$$\mathscr{S}_1 = \sum_{\theta \geqslant \lceil g/2 \rceil} 2y_\theta\Delta^{\theta-1} = \frac{4a}{\Delta^{\varepsilon}} \sum_{\theta \geqslant \lceil g/2 \rceil} \frac{1}{\gamma^{\theta}} \leqslant \frac{4a}{\Delta^{\varepsilon}\gamma^{\lceil g/2 \rceil-1}(\gamma-1)}$$

and

$$\mathscr{S}_2 = \sum_{\lambda \geqslant \lceil g/2 \rceil} z_\lambda\Delta^{2\lambda-2} = \frac{a^2}{\Delta^{2\varepsilon}} \sum_{\lambda \geqslant \lceil g/2 \rceil} \frac{1}{(\delta)^{2\lambda}} \leqslant \frac{a^2}{\Delta^{2\varepsilon}\delta^{2\lceil g/2 \rceil-2}(\delta^2-1)}.$$

Let $\mathscr{P}_i$, $\mathscr{N}_i$, and x_i denote, respectively, the probabilities, number of edges, and local lemma constants associated with events of type i. We can see that, as in the previous proof, the inequalities required by local lemma are satisfied provided that

$$\mathscr{P}_i \leqslant x_i\left(\frac{1}{4}\right)^{\mathscr{N}_i\Upsilon}, \quad \forall i, \tag{2}$$

where

$$\Upsilon = \frac{2}{\alpha\Delta^{\varepsilon}} + \frac{1}{\beta\Delta^{2\varepsilon}} + \frac{4a}{\Delta^{\varepsilon}\gamma^{\lceil g/2 \rceil-1}(\gamma-1)} + \frac{a^2}{\Delta^{2\varepsilon}\delta^{\lceil g/2 \rceil-2}(\delta^2-1)}.$$

By choosing c_1 suitably large, we can verify that $\Delta^{\varepsilon}\Upsilon \leqslant \frac{1}{125}$ and each of inequalities (2) is satisfied. As a result, the inequalities corresponding to local lemma are also satisfied. Finally, fixing $c_2 = a \cdot c_1$, the theorem is proved. $\square$

4. Remarks

We are able to bring down the upper bound on $a'(G)$ from 16Δ to 4.6Δ, assuming the girth to be at least a small constant. We believe that, with a more careful analysis, it will be possible to remove the girth assumption.

As we mentioned earlier, it is the short cycles which are difficult to deal with when we are using probabilistic arguments, because they have a higher probability of becoming bichromatic as compared to long cycles. Similarly, when we try to kill bichromatic cycles in a proper colouring by randomly recolouring some of the edges with a set of new colours, short cycles have a high probability of survival. We are presently investigating on how to take care of short cycles.

Appendix A. A note on the claimed 9Δ bound in [8]

The proof of $a'(G) \leqslant 9\Delta$ given in [8] is based on applying a specialised version of Lovász Local Lemma to the following random experiment: choose a colour for each edge independently and uniformly at random, from a set C of $a\Delta$ colours for some $a>1$. It is easy to see that the requirements of the local lemma are not met in the proof given.

We give below an argument explaining why the proof is not easily rectifiable even if we ignore the acyclicity requirements and only want to ensure properness. More precisely, we show that any proof, which is based on applying local lemma on the random experiment stated above and which assumes a uniform value for all the constants (associated with various events), will require that $a \geqslant 4e$. It is natural to assume that the constants are uniformly the same unless one wants to look at proofs which make use of the structure of the graph under consideration.

With respect to the random experiment, consider an unfavourable event that a pair of incident edges e, f receive the same colour. Denote it by $E_{e,f}$. Clearly, $\Pr(E_{e,f}) = 1/(a\Delta)$ and the number of other events which may influence a given event is at most 4Δ. Let x_0 be the uniform constant chosen for all events.

Applying the local lemma, we see that none of these bad events holds, if $1/(a\Delta) \leqslant x_0(1 - x_0)^{4\Delta}$. Write x_0 as $1/(\alpha\Delta)$. It follows that the inequality of the local lemma holds only if $1/a \leqslant (1/\alpha)(1 - 1/\alpha\Delta)^{4\Delta} \leqslant (1/\alpha)\mathrm{e}^{-4/\alpha}$. Let $f(\alpha) = (1/\alpha)\mathrm{e}^{-4/\alpha}$. To find the extrema, we have $f'(\alpha) = -(1/\alpha^2)\mathrm{e}^{-4/\alpha} + (4/\alpha^3)\mathrm{e}^{-4/\alpha} = 0$, which yields $\alpha^* = 4$. Since $f'(\alpha) = (4/\alpha^2 - 1/\alpha) f(\alpha)$, we get $f''(\alpha) = (4/\alpha^2 - 1/\alpha) f'(\alpha) + (1/\alpha^2 - 8/\alpha^3) f(\alpha)$. Since $f''(\alpha^*) = -1/(64e) < 0$, the maximum value of $f(\alpha)$ namely $1/(4e)$, is achieved at $\alpha = 4$. Hence we need to have $1/a \leqslant 1/(4e)$ or, equivalently, $a \geqslant 4e$.

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