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ORIENTED COLOURING OF SOME GRAPH PRODUCTS

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Abstract

We obtain some improved upper and lower bounds on the oriented chromatic number for different classes of products of graphs.

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1. Introduction

The concept of oriented colouring was introduced by Bruno Courcelle in [4]. Since then, many researchers have worked on the problem. It can be viewed as the 'natural' oriented version of ordinary undirected vertex colouring when one looks at it in terms of homomorphism [8]. The edge analogue of this has applications in wireless sensor networks [9]. Even for simple classes of graphs like 2-dimensional grid graphs or planar graphs, we do not know tight bounds on the associated chromatic number.

We first define what an oriented colouring is. An oriented graph $\overrightarrow{G} = (V, A)$ is an orientation of the edges of a simple undirected graph $G = (V, E)$. That is, $\overrightarrow{G}$ does not contain loops or opposite arcs. An oriented k-colouring of an oriented graph is a partition of its vertex set into k labelled subsets such that no two adjacent vertices belong to the same subset, and all the arcs between a pair of subsets have the same orientation. Precisely, an *oriented k-colouring* of an oriented graph $\overrightarrow{G}$ is a mapping $\mathcal{C} : V \mapsto [k]$ such that (i) $\mathcal{C}(x) \neq \mathcal{C}(y)$ for any arc $(x, y) \in A(\overrightarrow{G})$ and (ii) for all arcs (x, y) and (z, w) in $A(\overrightarrow{G})$, $\mathcal{C}(x) = \mathcal{C}(w)$ implies $\mathcal{C}(y) \neq \mathcal{C}(z)$. Notice that $\mathcal{C}(x)$ stands for the colour of the vertex x with respect to the colouring $\mathcal{C}$ and $[k]$ stands for the set $\{1, 2, \ldots, k\}$.

The *oriented chromatic number* of an oriented graph $\overrightarrow{G}$ is the smallest $k \in \mathbb{N}$ that admits an oriented vertex k-colouring of $\overrightarrow{G}$ and is denoted by $\chi_o(\overrightarrow{G})$. One can also view an oriented k-colouring as a homomorphism from $\overrightarrow{G}$ to a suitable oriented graph on k vertices. A *homomorphism* from a directed graph $\overrightarrow{G}$ to a directed graph $\overrightarrow{H}$ is a mapping that preserves the arcs. That is, $\phi : V(\overrightarrow{G}) \mapsto V(\overrightarrow{H})$ is a homomorphism if $(\phi(u), \phi(v)) \in A(\overrightarrow{H})$ for every arc (u, v) in $A(\overrightarrow{G})$. Hence we note that $\chi_o(\overrightarrow{G})$ is the smallest order of an oriented graph $\overrightarrow{H}$ such that there is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. The oriented chromatic number of an undirected graph G, denoted $\chi_o(G)$, is the maximum of $\chi_o(\overrightarrow{G})$ taken over all orientations $\overrightarrow{G}$ of G.

Bounds for the oriented chromatic number have been obtained in terms of the maximum degree [11] as well as for various special families of graphs such as trees, partial k-trees [17], 1-planar graphs [3] and graphs of bounded genus [1]. As mentioned earlier, the oriented chromatic number of planar graphs is unknown and an important open problem in this area. The best known lower bound is 17 [12] while the best upper bound proved so far is 80 [16]. By considering subclasses of planar graphs, we might expect to prove improved and possibly tight bounds. In this direction, bounds have been obtained for triangle-free planar graphs [14], for planar graphs with girth restrictions [13], for outerplanar graphs [15], 2-outerplanar graphs [5] and for grid graphs [6]. The last-mentioned result uses the "product" structure of grids and suggests the natural problem of obtaining bounds for products of graphs in terms of the oriented chromatic number of the individual graphs. Our work in this paper is a step in this direction.

For an excellent survey of oriented colouring, see [18]. In this paper, we obtain bounds on the oriented chromatic number of Cartesian products and strong products of arbitrary graphs with paths.

We now start with a few important definitions and then proceed to present the results we have obtained.

2. DEFINITIONS AND RESULTS

For any graph G (directed or undirected), we use $|G|$ to denote $|V(G)|$, the order of G.

An automorphism of an oriented graph $\overrightarrow{G}$ is a bijection from $V(\overrightarrow{G})$ to itself that preserves edges, non-edges and directions of the edges. If an automorphism does not map any vertex to itself, we call it a *non-fixing automorphism*.

In the following, K_2 denotes an edge and P_k denotes an undirected path on k vertices. The Cartesian and strong product of graphs are defined as follows.

Definition 2.1 Cartesian Product**.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Their *Cartesian product*, denoted $G_1 \square G_2$ is the graph (V, E) where $V = V_1 \times V_2$ and $([u_1, u_2], [v_1, v_2]) \in E$ if and only if **either** $u_1 = v_1$ and $(u_2, v_2) \in E_2$ **or** $u_2 = v_2$ and $(u_1, v_1) \in E_1$.

Definition 2.2 Strong Product**.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Their *strong product* denoted $G_1 \boxtimes G_2$ is the graph (V, E) where $V = V_1 \times V_2$ and $([u_1, u_2], [v_1, v_2]) \in E$ if and only if **either** $u_1 = v_1$ and $(u_2, v_2) \in E_2$ **or** $u_2 = v_2$ and $(u_1, v_1) \in E_1$ **or** $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$.

The Cartesian and strong products are both associative and commutative up to isomorphism. Furthermore, there is a unique prime factorisation (UPF) of any connected graph into a Cartesian (strong) product of prime graphs [10]. The *directed Cartesian product* and *directed strong product* are defined analogously with the edge set being replaced by the arc set.

For any graph G, let $G^n_\square$ and $G^n_\boxtimes$ denote respectively the Cartesian and strong products of G with itself n times. The graph $\mathcal{H}_d = {K_2}^d_\square$ is called the *hypercube* of dimension d. In other words $\mathcal{H}_d$ is the Cartesian product of d edges. The *d-dimensional hypergrid (mesh)* denoted $\mathcal{M}_d$ is the Cartesian product of d paths. The graph $\mathcal{M}_{m,n} = P_m \,\square\, P_n$ is called an *$m \times n$ grid*. We call the graph $\mathcal{S}_{m,n} = P_m \boxtimes P_n$ an *$m \times n$ strong-grid*.

A tournament is an orientation of an undirected complete graph. Let n be a prime number of the form $4k+3$. Let $c_1, c_2, \ldots, c_d$ be the non-zero quadratic residues of n. It is known that $d = \frac{n-1}{2}$. Define the directed graph $\vec{T}_n = T(n; c_1, \ldots, c_d)$ over $V = \{0, 1, \ldots, n-1\}$ as follows. For every $x, y \in V$, $x \neq y$, (x, y) is an arc if and only if $y = x + c_i$ for some $i \in [d]$. It is well-known that $\vec{T}_n$ is a tournament and is called the *Paley tournament* of order n.

A graph G is *arc transitive* if for any two arcs e, f in G, there exists an automorphism mapping e to f. In other words an arc-transitive graph is a graph such that any two arcs are equivalent under some element of its automorphism group. It is a well-known fact [7] that Paley tournaments are arc transitive.

We obtain the following results on $\chi_o(G)$ when G is a product of undirected graphs or oriented graphs. We also propose a conjecture. Specifically, we prove the following.

Theorem 1. *Let $\vec{G}$ be an oriented graph and $\vec{T}$ be a Paley tournament such that $\chi_o(\vec{G}) = |\vec{T}|$. Let $\vec{P}_k$ be any orientation of P_k. Assume that there is a homomorphism $\phi : V(\vec{G}) \mapsto V(\vec{T})$. Then $\chi_o(\vec{G} \,\square\, \vec{P}_k) = \chi_o(\vec{G})$, $k \geq 1$.*

The smallest Paley tournament is the directed cycle on three vertices. The Cartesian product of two edges contains directed 2-paths and thus requires 3 colours. Thus, we get the following result as a corollary to the above theorem,

Corollary 1.1. For the oriented product $\vec{\mathcal{H}}_d$ of d oriented edges, we have $\chi_o(\vec{\mathcal{H}}_d) = 3$.

The next result is for undirected Cartesian product of any graph with paths or cycles.

Theorem 2. *For any undirected graph G,*

1. $\chi_o(G \,\square\, P_k) \leq (2k-1)\chi_o(G)$, $k \geq 1$.
2. $\chi_o(G \,\square\, C_k) \leq 2k\chi_o(G)$, $k \geq 3$.

We also obtain the following for the strong product of paths.

Theorem 3. *For the strong product of undirected paths, we have the following.*

1. $8 \le \chi_o(\mathcal{S}_{2,n}) \le 11$,
2. $10 \le \chi_o(\mathcal{S}_{3,n}) \le 67$.

We believe and conjecture that Theorem 1 above can be strengthened to the following.

Conjecture 2.1. Let $\overrightarrow{H}$ be an arc transitive oriented graph having a non-fixing automorphism. Then, if $\phi : V(G) \mapsto V(H)$ is a homomorphism such that $\chi_o(\overrightarrow{G}) = |\overrightarrow{H}|$, then $\chi_o(\overrightarrow{G} \,\square\, \overrightarrow{P}_k) = \chi_o(\overrightarrow{G})$.

3. Proofs

In this section we prove the various results claimed above. First, we note that when $k = 1$ in either of the possible cases, there is nothing to prove as the cases follow trivially. We start with the proof of Theorem 1.

3.1. Proof of Theorem 1

Proof. Let $\overrightarrow{G}$ be an oriented graph and $\overrightarrow{T}$ be a Paley tournament satisfying the conditions of the theorem. Now consider the automorphism $\pi(i) = i + 1 \bmod p$, where p is the order of the Paley tournament. It is not difficult to see that π and hence π^{-1} are both non-fixing automorphisms. From the definition of π and the fact that 1 is a quadratic residue of p, it follows that $(u, \pi(u)) \in A(\overrightarrow{T})$ for every $u \in V(\overrightarrow{T})$.

Now, we colour the graph $\overrightarrow{G} \,\square\, \overrightarrow{P}_k$ as follows. Let $\overrightarrow{G}_i$, $i = 0, 1, \ldots, k-1$, be the i^{th} copy of $\overrightarrow{G}$ in $\overrightarrow{G} \square \overrightarrow{P}_k$. We colour inductively in the order $\overrightarrow{G}_0, \overrightarrow{G}_1, \ldots$. We colour the copy $\overrightarrow{G}_0$ with the homomorphism ϕ. To colour $\overrightarrow{G}_i$, $i \ge 1$, consider the orientation of the arcs between $\overrightarrow{G}_{i-1}$ and $\overrightarrow{G}_i$. If they are from $\overrightarrow{G}_{i-1}$ to $\overrightarrow{G}_i$, each vertex $x \in \overrightarrow{G}_i$ is coloured with $\pi(c_x)$ where c_x is the colour of x in $\overrightarrow{G}_{i-1}$. On the other hand, if the arcs are from $\overrightarrow{G}_i$ to $\overrightarrow{G}_{i-1}$, each vertex $x \in \overrightarrow{G}_i$ is coloured with $\pi^{-1}(c_x)$.

We claim that the above colouring is an oriented colouring of the graph $\overrightarrow{G} \square \overrightarrow{P}_k$. Each $\overrightarrow{G}_i$ mapped to $\overrightarrow{T}$ by the homomorphism $\sigma_{i-1}\sigma_{i-2} \ldots \sigma_1 \phi$ where each σ_i, $i \ge 1$ is either π or π^{-1} depending on the orientation of $\overrightarrow{P}_k$. For any vertex u in $\overrightarrow{G}$, let u_i and c_{u_i} denote the vertex corresponding to u and its colour within $\vec{G}_i$. If $(i, i+1) \in A(\vec{P}_k)$, then in our colouring, $(c_{u_i}, c_{u_{i+1}}) \in A(\vec{T})$. Similarly, if $(i+1, i) \in A(\vec{P}_k)$, then we have $(c_{u_{i+1}}, c_{u_i}) \in A(\vec{T})$. Thus, we have extended the homomorphism $\phi : \vec{G} \mapsto \vec{T}$ to a homomorphism from $\vec{G} \,\square\, \vec{P}_k$ into $\vec{T}$. Since $\vec{G}$ is a subgraph of $\vec{G} \,\square\, \vec{P}_k$, it follows that $\chi_o(\vec{G} \,\square\, \vec{P}_k) = \chi_o(\vec{G})$. ■

3.2. Proof of Theorem 2

Proof. We now prove that $\chi_o(G \,\square\, P_k) \leq (2k-1)\chi_o(G)$. Fix any arbitrary orientation of the product. By definition, any orientation $\vec{G}$ of G can be coloured with $\chi_o(G)$ colours. For each of the k oriented (perhaps differently) copies $\vec{G}_0, \vec{G}_1, \ldots, \vec{G}_{k-1}$ of G in $G \,\square\, P_k$, we initially colour the vertices of $\vec{G}_i$ using a distinct set of $\chi_o(G)$ colours. Now starting from $\vec{G}_0$, we inductively recolour each $\vec{G}_i$ as follows. To colour $\vec{G}_i$, consider the copies $\vec{G}_i$ and $\vec{G}_{i+1}$. For each colour c used in $\vec{G}_i$, consider the set $C_0(c)$ of vertices coloured c in $\vec{G}_i$ which has arcs going to $\vec{G}_{i+1}$ and the set $C_1(c)$ of vertices coloured c having arcs coming from $\vec{G}_{i+1}$. Now we split the colour class corresponding to c into c_0 and c_1. We repeat this for every colour in $\vec{G}_i$. Notice that this ensures that there are no pair of colours c (used in $\vec{G}_i$) and d (used in $\vec{G}_{i+1}$) having arcs in both directions between colour classes of c and d. Thus, we have used at most $2\chi_o(G)$ colours in the copy $\vec{G}_i$. We perform this operation on each $\vec{G}_i$, $0 \leq i < k-1$. Note that splitting the colour classes of $\vec{G}_i$ does not introduce violations w.r.t. edges between $\vec{G}_{i-1}$ and $\vec{G}_i$. Thus, we have used at most $(2k-1)\chi_o(G)$ colours. It is easily seen that the entire colouring is oriented and proper.

Notice that the above argument can be easily and directly extended to the case of products with cycles as well except that we need to perform the doubling in all the k copies. Thus, for any graph G, $\chi_o(G \,\square\, C_k) \leq 2k\chi_o(G)$. ■

3.3. Proof of Theorem 3

A property of oriented colouring that we mentioned in Section 2 and use repeatedly in our arguments is that directed 2-paths need distinct colours on all three vertices. If u-v-w is a directed path from u to w, then in any oriented colouring, the colours of u,v and w are distinct.

Proof. First we show that $8 \leq \chi_o(\mathcal{S}_{2,n}) \leq 11$. We obtain the upper bound by establishing the existence of a homomorphism from any orientation $\vec{\mathcal{S}}_{2,n}$ of $\mathcal{S}_{2,n}$ into the Paley tournament $\vec{T}_{11}$.

Consider the Figure 1. We map the vertices of $\vec{\mathcal{S}}_{2,n}$ to $\vec{T}_{11}$ inductively. We assume that the vertices are coloured from the left up to and including the vertices x and y. Now we show that, for all possible orientations of the dotted arcs across, we can extend the partial homomorphism (colouring) to the vertices a and b.

We make use of the fact mentioned earlier that the Paley tournament $\vec{T}_{11}$ is arc transitive. Hence we may assume, without loss of generality, that the vertices x and y are coloured with 0 and 1 respectively.

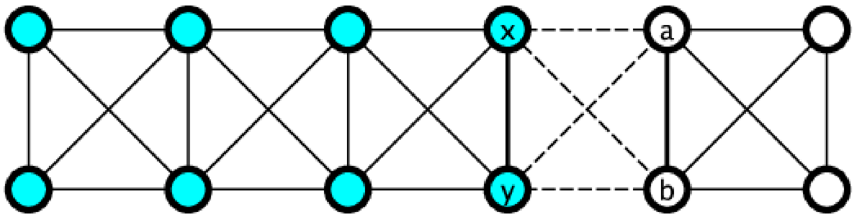

Figure 1. The partially coloured $2 \times n$ grid.

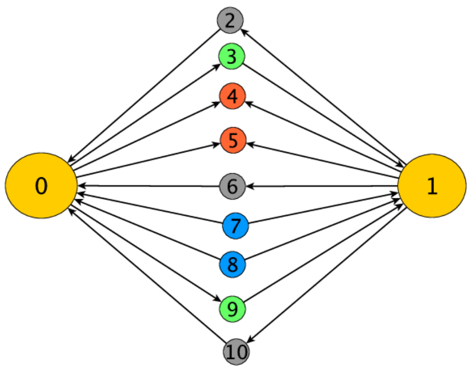

Figure 2. Orientations of 2-ears in T_{11}.

Consider the dotted 2-paths x-a-y and x-b-y. We have the following two cases.

Case 1. The 2-paths x-a-y and x-b-y are identically oriented. We can see from Figure 2 that for every possible orientation of a 2-path, there are at least 2 vertices p and q such that the ears 0-p-1 and 0-q-1 are identically oriented. Now we colour a and b suitably by looking at the orientations of the arcs between a and b and between p and q.

Case 2. The orientations of x-a-y and x-b-y are not the same. Now we notice that we have at least 2 possible colours (say $\{r, s\}$) satisfying the orientation of x-a-y as well as a disjoint set of 2-colours (say $\{t, u\}$) for x-b-y.

We show that between $\{r, s\}$ and $\{t, u\}$, there is at least one arc which satisfies the orientation of (a, b). Recall that we assumed that x and y are respectively coloured 0 and 1. The sets of colours having the same orientation with respect to 0 and 1 are $\{4, 5\}$, $\{3, 9\}$, $\{7, 8\}$, and $\{2, 6, 10\}$. The following table shows that for each pair of these sets, there are valid arcs between them in both directions (we provide one for each direction) proving the claim. (For example, the entry $(3, 7)$ in the table shows that an arc from 3 to 7 is valid corresponding to the pairs of sets $\{3, 9\}$ and $\{7, 8\}$ having the same colour.

Table 1. Table showing the pairs of arcs between each pair in both directions.

	4,5	3,9	7,8	2,6,10
4,5	.	(5,9)	(4,8)	(4,2)
3,9	(3,4)	.	(3,7)	(9,2)
7,8	(7,5)	(8,9)	.	(8,2)
2,6,10	(6,4)	(2,3)	(2,7)	.

Hence we can inductively extend the colouring to $\vec{\mathcal{S}}_{2,n}$.

The lower bound is explicit from the oriented graph depicted in Figure 3 which requires 8 colours in any oriented colouring. To see this, notice that the orientation is such that any 2×3 block of 6 vertices need to be given distinct colours (because of oriented 3-paths and adjacency). Hence the first 6 vertices get 6 distinct colours.

Due to the same reason, the next two vertices (red vertices) cannot be given colours 3 or 4. Hence there are two possibilities. The first one is to reuse at least one of the colours 1 or 2 on one of the red vertices. If either of the red vertices are coloured 1 or 2, the last two vertices (grey vertices) cannot get 3 or 4 since the orientation of arcs are already fixed to be from

$\{1,2\}$ to $\{3,4\}$ and we need to introduce two new colours (say 7 and 8). In the other case, we need to colour the red vertices by 2 new colours. Thus, in either case it requires 8 colours.

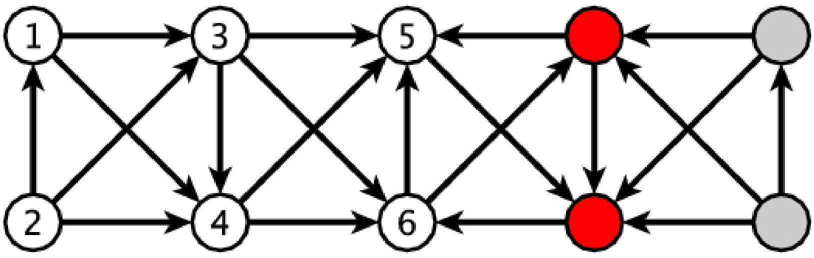

Figure 3. A 2×5 oriented grid requiring 8 colours.

3.3.1. The second result

We now prove that $10 \leq \chi_o(\mathcal{S}_{3,n}) \leq 67$. Here we have a huge gap between the upper and lower bounds. Once again, we map the vertices to $\vec{T}_{67}$ to show the upper bound. The lower bound follows from the fact that, many orientations of $\mathcal{S}_{3,5}$ (e.g. Figure 4) requires at least 10 colours.

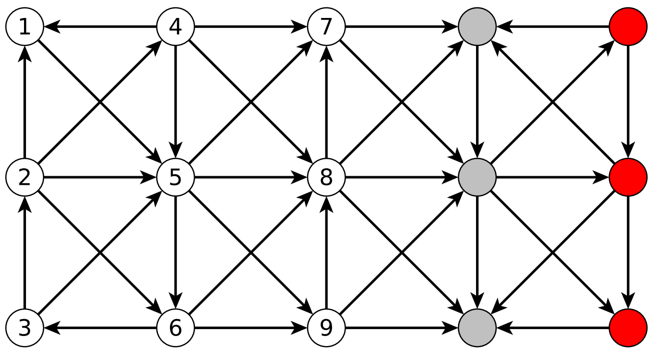

Figure 4. An orientation of $\mathcal{S}_{3,5}$ that requires 10 colours.

To check this, notice first that the leftmost 3×3 block must use 9 distinct colours. Assume now that we only use 9 colours for the whole graph. Notice that only colours 1, 2 and 3 can appear on the grey vertices and thus 4, 5 and 6 on the red vertices. For any colouring of the grey vertices with 1, 2 and 3, colour 5 cannot appear on any of the red vertices since all the arcs from 1, 2 and 3 need to be towards 5 (to preserve the orientation), and each red vertex has at least one outgoing arrow to one of the grey vertices. Therefore 10 colours are necessary.

Now we prove the upper bound.

Let $\vec{G}$ be any orientation of $\mathcal{S}_{3,n}$. As before, we construct a homomorphism from $\vec{G}$ to a Paley tournament, namely $\vec{T}_{67}$. We first state some definitions. An orientation vector of size m is a sequence $\alpha = (\alpha_1, \ldots, \alpha_m)$ in $\{0,1\}^m$. Given a sequence $X = (x_1, \ldots, x_m)$ of vertices in an oriented graph $\vec{G}$, an α-successor of X is a vertex y such that for each i, $(x_i, y) \in A(\vec{G})$ if $\alpha_i = 1$ and $(y, x_i) \in A(\vec{G})$ if $\alpha_i = 0$. We say that an oriented graph has Property $P(m,k)$ if for any sequence X of m distinct vertices in the graph and for any orientation vector α in $\{0,1\}^m$, there are at least k α-successors of X. Note that the property $P(m,k)$ implies the property $P(n,k)$ for all $n < m$. Paley tournaments are good candidates for satisfying such properties and in particular, we shall use the following fact:

Fact 3.1. $\vec{T}_{67}$ satisfies Property $P(4,1)$ [5] as well as Property $P(2,2)$ [2].

We now map $\mathcal{S}_{3,n}$ to $\vec{T}_{67}$ using induction on n.

Base case: $n = 1$. In this case, $\vec{G}$ is just an oriented path on 3 vertices and it is easy to map the 3 vertices of $\vec{G}$ to 3 distinct vertices in $\vec{T}_{67}$ using Property $P(2,2)$.

Induction step: Assume that the subgraph induced by the vertices $[i,j] : i \in \{0,1,2\}, j \in \{0,1,\ldots,n-2\}$ are mapped to $\vec{T}_{67}$. Now $[0,n-1]$ has exactly 2 neighbours which have already been mapped (coloured) to distinct vertices in $\vec{T}_{67}$ and since $\vec{T}_{67}$ has Property $P(2,2)$, we can extend the mapping so that $[0,n-1]$ is mapped to a vertex in $\vec{T}_{67}$ that is different from the image of $[2,n-2]$. We now see that $[1,n-1]$ has four neighbours that have already been mapped to four distinct vertices in $\vec{T}_{67}$ and using the property $P(4,1)$ of $\vec{T}_{67}$, we extend the mapping to the vertex $[1,n-1]$. We can now map $[2,n-1]$ to $\vec{T}_{67}$ as well since it has three distinct coloured neighbours and we make use of the property $P(3,1)$ of $\vec{T}_{67}$. This completes the proof of part (ii) of Theorem 3. ∎

4. Conclusions

We obtained bounds on oriented chromatic numbers for products of arbitrary graphs with paths. However, there are gaps between the lower and upper bounds and we leave open the problem of narrowing them. In particular, obtaining the exact value of $\chi_o(\mathcal{S}_{m,n})$ is an interesting problem.

References

[1] N.R. Aravind and C.R. Subramanian, *Forbidden subgraph colorings and the oriented chromatic number*, in: 4th International Workshop on Combinatorial Algorithms **4393** (2009) 477–488.

[2] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud and É. Sopena, *On the maximum average degree and the oriented chromatic number of a graph*, Discrete Math. **206** (1999) 77–89.

[3] O.V. Borodin, A.V. Kostochka, A. Raspaud and É. Sopena. *Acyclic colouring of* 1*-planar graphs*, Discrete Appl. Math. **114** (2001) 29–41.

[4] B. Courcelle, *The monadic second order logic of graphs. vi. on several representations of graphs by relational structures*, Discrete Appl. Math. **54** (1994) 117–149.

[5] L. Esperet and P. Ochem, *Oriented colorings of* 2*-outerplanar graphs*, Information Processing Letters **101** (2007) 215–219.

[6] G. Fertin, A. Raspaud and A. Roychowdhury, *On the oriented chromatic number of grids*, Information Processing Letters **85** (2003) 261–266.

[7] E. Fried, *On homogeneous tournaments*, Combinatorial Theory and its Applications **2** (1970) 467–476.

[8] P. Hell and J. Nešetřil, Graphs and Homomorphisms, Oxford Lecture Series in Mathematics and its Applications (28), 2004.

[9] T. Herman, I. Pirwani and S. Pemmaraju, *Oriented edge colorings and link scheduling in sensor networks*, in: SENSORWARE 2006: 1st International Workshop on Software for Sensor Networks, 2006.

[10] W. Imrich and S. Klavžar, Product Graphs : Structure and Recognition (John Wiley & Sons, Inc., 2000).

[11] A.V. Kostochka, É. Sopena and X. Zhu, *Acyclic and oriented chromatic numbers of graphs*, J. Graph Theory **24** (1997) 331–340.

[12] T.H. Marshall, *Oriented colorings of triangle-free planar graphs*, J. Graph Theory **55** (2007) 175–190.

[13] J. Nešetřil, A. Raspaud and É. Sopena, *Colorings and girth of oriented planar graphs*, Discrete Math. **165–166** (1997) 519–530.

[14] P. Ochem, *Oriented colorings of triangle-free planar graphs*, Information Processing Letters **92** (2004) 71–76.

[15] A. Pinlou and É. Sopena, *Oriented vertex and arc colorings of outerplanar graphs*, Information Processing Letters **100** (2006) 97–104.

[16] A. Raspaud and É. Sopena, *Good and semi-strong colorings of oriented planar graphs*, Information Processing Letters **51** (1994) 171–174.

[17] É. Sopena, *The chromatic number of oriented graphs*, J. Graph Theory **25** (1997) 191–205.

[18] É Sopena, *Oriented graph coloring*, Discrete Math. **229** (2001) 359–369.

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