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Improved bounds on acyclic edge colouring

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Abstract

A *proper* colouring of the edges of a graph G is called *acyclic* if there is no two-coloured cycle in G. The *acyclic chromatic index* of G, denoted $a'(G)$, is the least number of colours required for an acyclic edge colouring of G. It is known that $a'(G) \leq 16\Delta$ for all graphs G where Δ is the maximum degree of G. We prove that $a'(G) \leq 6\Delta$ for all graphs with girth, $g(G) \geq 9$ and also that $a'(G) \leq 4.52\Delta$ if $g(G) \geq 220$. We then derive a relationship between g and $a'(G)$.

Keywords: acyclic edge colouring, graphs, acyclic chromatic index, girth

1 Introduction

We consider only simple undirected graphs. Throughout the paper, we use $\Delta = \Delta(G)$ to denote the maximum degree of a graph G and $g = g(G)$ to denote the girth (length of a shortest cycle) of G. A colouring of the edges of a graph is *proper* if no pair of incident edges receive the same colour. A proper colouring of the edges of a graph G is called *acyclic* if there is no bichromatic (two-coloured) cycle in G. In other words, the subgraph induced by the union of any two colour classes is a forest. The minimum number of colours required

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for any acyclic edge colouring is called the *acyclic chromatic index* of G and is denoted by $a'(G)$. This notion was introduced by Grunbaum in [5].

It is obvious that any proper edge colouring of G requires at least Δ colours, and Vizing [8] showed that there exists a proper edge colouring with $\Delta + 1$ colours. Alon et.al [2] obtained a bound of 64Δ for $a'(G)$ which was later improved to 16Δ (best known upperbound) in [6]. In [7] (Ch.19, page 226), a bound of 9Δ is claimed for $a'(G)$ but the proof is incorrect and is not rectifiable (see Appendix A for details.

In this work we obtain a bound of 4.52Δ for all graphs with $g(G) \geq 220$. We can relax the girth requirement to 9 if we are willing to use 6Δ colours. It might be possible to remove the girth requirement with a more sophisticated analysis.

Theorem 1.1 *If $g(G) \geq 9$, then $a'(G) \leq 6\Delta$.*

Theorem 1.2 *If $g(G) \geq 220$, then $a'(G) \leq 4.52\Delta$.*

The main idea of the proof is to allow limited improperness with acyclicity in the first phase and take care of properness in the second phase. Both theorems are proved using essentially the same arguments.

It is conjectured that $a'(G) \leq \Delta + 2$ in [1] and in the same work this conjecture is proved for graphs with $g = \Omega(\Delta \log \Delta)$ and also that $a'(G) \leq 2\Delta + 2$ for graphs with $g = \Omega(\log \Delta)$. However, when using probabilistic arguments, short cycles are the major obstacles since they have a “high” probability of becoming bichromatic as compared to long cycles. Hence bounding $a'(G)$ for graphs with short cycles seems fairly difficult. The following theorem tries to capture this phenomenon in a formal way. As a corollary, we notice that $a'(G) \leq \Delta + o(\Delta)$ for all graphs G with $g = \omega(\log \Delta)$. All logarithms are to the base e.

Theorem 1.3 *There are absolute constants c_1, $c_2 > 0$ such that, for any G with $g \geq c_1 \log \Delta$ we have, $a'(G) \leq \Delta + 1 + \left\lceil c_2 \left(\frac{\Delta \log \Delta}{g}\right)\right\rceil$*

Our proofs are based on probabilistic arguments and we make use of Lovàsz Local Lemma (general version, see [4,7] for details). In Section 2, we prove Theorems 1.1 and 1.2. In Section 3, we prove Theorem 1.3.

2 Proof of theorems 1.1 and 1.2

We have not optimised the constants in the application of Local Lemma. With a more careful application of Local Lemma it might be possible to bring down the bound a bit further.

It is known that when $\Delta \leq 3$, $a'(G) \leq \Delta + 2$ [3]. Hence we may assume that $\Delta \geq 4$ in our arguments. Our proof consists of two stages. In the first stage, we show, by probabilistic arguments, the existence of a colouring $\mathcal{C}$, using a set C of $c\Delta$ colours (where $c > 1$ is a constant to be fixed later), such that $\mathcal{C}$ satisfies the following properties for some positive integer $\eta \leq 4$.

(i) every vertex has at most η incident edges of any single colour,

(ii) there are no properly two-coloured cycles, and

(iii) there are no monochromatically coloured cycles.

Note that in $\mathcal{C}$, each *colour class* (set of edges receiving the same colour) is a forest of maximum degree at most η. In the second stage we split each such forest into η matchings by recolouring the edges of each colour a_i with the colours $a_i^1, \ldots, a_i^\eta$ to get a colouring $\mathcal{C}'$. It is easy to see that $\mathcal{C}'$ is proper, acyclic and uses at most $c\eta\Delta$ colours.

To complete the proof, it is now sufficient to show that such a colouring $\mathcal{C}$ described above exists with positive probability, which we do using Local Lemma. We do the following random experiment. Each edge chooses a colour uniformly and independently at random from the set C. For the resulting random colouring to satisfy (i)-(iii) above, define the following three types of *unfavourable* events classified into types (1)-(3).

(i) For a set of $\eta+1$ edges $\{e_1, \ldots, e_{\eta+1}\}$ incident on a vertex u, let $E_{e_1,\ldots,e_{\eta+1}}$ be the event that all of them receive the same colour.

(ii) Let $E_{D,2k}$ denote the event that an even cycle D of length $2k$ is properly coloured with 2 colours.

(iii) Let $E_{D,\ell}$ denote the event that a cycle D of length ℓ is coloured monochromatically.

It is easy to verify that, if none of the above events hold, then $\mathcal{C}$ satisfies (i)-(iii). In order to apply the Local Lemma, we need estimates for the probabilities of each event, and also for the number of other events of each type, which can possibly influence any given event. For this purpose, we calculate the number of events of each type that depend on a given edge, and multiply by the number of edges to get an upper bound on the dependency (an event influences another only if they share an edge). We present only the estimates whose proof uses standard arguments [2].

The probabilities of events (1)-(3) are at most, respectively, $\frac{1}{|C|^\eta}$, $\frac{1}{|C|^{2k-2}}$ and $\frac{1}{|C|^{\ell-1}}$. Also for any given edge e less than, $\frac{2\Delta^\eta}{\eta!}$ type (1), Δ^{2k-2} type (2) and $\Delta^{\ell-2}$ type (3) events depend on e.

In order to apply Local Lemma, let $x_0 = 1/(\alpha\Delta)^\eta$, $x_k = 1/(\beta\Delta)^{2k-2}$ and $y_\ell = 1/(\gamma\Delta)^{\ell-1}$, be the values associated with events of types (1)-(3) respectively, where α,β,$\gamma > 1$ are constants to be determined later. Recall that we use g to denote girth. We conclude that, with positive probability none of the above events occur, provided $\forall k \geq \lceil \frac{g}{2} \rceil, \ell \geq g$

$$\frac{1}{(c\Delta)^\eta} \leq x_0 (1-x_0)^{(\eta+1)\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-x_\theta)^{(\eta+1)\Delta^{2\theta-2}} \prod_{\lambda \geq g} (1-y_\lambda)^{(\eta+1)\Delta^{\lambda-2}}$$

$$\frac{1}{(c\Delta)^{2k-2}} \leq x_k (1-x_0)^{2k\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-x_\theta)^{2k\Delta^{2\theta-2}} \prod_{\lambda \geq g} (1-y_\lambda)^{2k\Delta^{\lambda-2}}$$

$$\frac{1}{(c\Delta)^{\ell-1}} \leq y_\ell (1-x_0)^{\ell\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-x_\theta)^{\ell\Delta^{2\theta-2}} \prod_{\lambda \geq g} (1-y_\lambda)^{\ell\Delta^{\lambda-2}}$$

Let $f(z) = (1-\frac{1}{z})^z$. It is well-known that $(1-\frac{1}{z})^z \uparrow \frac{1}{\mathrm{e}}$. Define

$$\Lambda = \min\left\{ f({x_0}^{-1}),\ \min_{\theta \geq \lceil \frac{g}{2} \rceil} f({x_\theta}^{-1}),\ \min_{\lambda \geq g} f({y_\lambda}^{-1}) \right\}.$$

The three inequalities required by Local Lemma are satisfied $\forall k \geq \lceil \frac{g}{2} \rceil$, $\ell \geq g$ (see Appendix B), provided

$$(1) \qquad \frac{1}{c} \leq \frac{1}{\alpha}\Lambda^{\frac{\eta+1}{\eta}\Upsilon}, \qquad \frac{1}{c} \leq \frac{1}{\beta}\Lambda^{\frac{2k}{2k-2}\Upsilon} \quad \text{and} \quad \frac{1}{c} \leq \frac{1}{\gamma}\Lambda^{\frac{\ell}{\ell-1}\Upsilon}$$

$$\text{where} \quad \Upsilon = \frac{2}{\eta!\alpha^\eta} + \frac{1}{(\beta^2-1)\beta^{2\lceil \frac{g}{2} \rceil - 4}} + \frac{1}{\Delta\gamma^{g-2}(\gamma-1)}.$$

Now we have to set specific values of α, β, γ and η. First we set $\eta = 2$ and $\alpha = \beta = \gamma = 2$. Using $g \geq 9$ and $\Delta \geq 4$, we have $\Lambda \geq (1-\frac{1}{64})^{64} \geq 0.3649$. It can easily be verified that that the above inequalities (1) are satisfied by setting $c = 2.951$. It follows that $a'(G) \leq 5.91\Delta < 6\Delta$ for all graphs G with girth $g \geq 9$. This proves Theorem 1.1.

Secondly, we set $\eta = 4$, $\alpha = 1.02$, $\beta = 1.04$ and $\gamma = 1.04$. Using $g \geq 220$ and $\Delta \geq 4$, we have $\Lambda \geq (1-\frac{1}{256})^{256} \geq 0.3671$. It follows that by setting $c = 1.13$, $a'(G) \leq 4 \times 1.13\Delta = 4.52\Delta$ when girth $g \geq 220$. Hence Theorem 1.2. □

3 Proof of theorem 1.3

An even cycle is called *half-monochromatic* with respect to a colouring if some *half* (a set of alternate edges) of it is monochromatic. Notice that this means every bichromatic cycle is also half-monochromatic.

For the sake of simplicity in the analysis, we write g in the form $c_1\Delta^\varepsilon \log\Delta$, where $\varepsilon \geq 0$ and where c_1 is mentioned in Theorem 1.3. We can assume $w.l.o.g.$ that $\varepsilon \leq 1$, because when ε exceeds 1, by choosing a large value of c_1, $a'(G) \leq \Delta + 2$ as shown in [1]. As before, we assume $\Delta \geq 4$.

The proof consists of an initial deterministic phase followed by a random phase. We begin by obtaining a proper edge colouring of G using $\Delta+1$ colours by Vizing's method. We then randomly recolour some of the edges with a new set of colours and show that with positive probability the colouring obtained is proper and acyclic. The reandom experiment is a modification of the one used in [1]. The random colouring is obtained as follows:

(i) Obtain a proper colouring $\mathcal{C} : E \rightarrow S_1 = \{1, \ldots, \Delta + 1\}$.

(ii) In the second phase we do the following:
- Activate each edge with independent probability $p = \frac{1}{\Delta^{\varepsilon}}$.
- Each activated edge chooses a new colour uniformly at random and independently, from the set $S_2 = \{1', \ldots, (a\Delta^{1-\varepsilon})'\}$, where $a > 1$ is a constant to be determined later.

Denote the resulting random colouring by $\mathcal{C}'$. With respect to $\mathcal{C}'$, we define the following *unfavourable* events classified into types (1)-(4).

(i) For a pair of incident edges e and f, let $E_{e,f}$ denote the event that they are both recoloured with the same new colour.

(ii) Let $E_{C,2k}$ denote the event that a bichromatic cycle C of length $2k$ in $\mathcal{C}$ is undisturbed in the recolouring process.

(iii) Let $E_{C,2\ell}$ denote the event that a half-monochromatic cycle C of length 2ℓ in $\mathcal{C}$ becomes bichromatic by retaining the same colour on a half and receiving a common new colour on the other half.

(iv) Let $E_{C,2m}$ denote the event where an even length cycle, C of length $2m$ becomes properly bichromatic with 2 of the new colours.

We can verify that, the absence of type (1)-(4) events imply that the colouring $\mathcal{C}'$ is proper and is also acyclic. It is therefore sufficient to show the absence of the above four types of events.

Upper bounds on probabilities for events of type (1)-(4) are given by $\frac{1}{a\Delta^{1+\varepsilon}}$, $e^{\frac{-2k}{\Delta^{\varepsilon}}}$, $\frac{2a\Delta^{1-\varepsilon}}{(a\Delta)^{\ell}}$ and $\frac{(a\Delta^{1-\varepsilon})^2}{(a\Delta)^{2m}}$. Also, for any given edge e, less than 2Δ type (1), Δ type (2), $2\Delta^{\ell-1}$ type (3) and Δ^{2m-2} type (4) events influence e. To apply Local Lemma, let $x_0 = 1/(\alpha\Delta^{1+\varepsilon})$, $x_1 = 1/(\beta\Delta^{1+2\varepsilon})$, $y_\ell = (2a\Delta^{1-\varepsilon})/(\gamma\Delta)^{\ell}$ and $z_m = (a\Delta^{1-\varepsilon})^2/((\delta\Delta)^{2m})$ be the values associated with events of type (1)-(4) respectively. Here $\alpha, \beta, \gamma, \delta > 1$ are real values to be determined later. We conclude that, with positive probability none of the above events occur, provided $\forall k, \ell, m \geq \lceil \frac{g}{2} \rceil$,

$$\frac{1}{a\Delta^{1+\varepsilon}} \leq x_0(1-x_0)^{4\Delta}(1-x_1)^{2\Delta} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4\Delta^{\theta-1}} \prod_{\lambda \geq \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2\Delta^{2\lambda-2}}$$

$$e^{\frac{-2k}{\Delta^\varepsilon}} \leq x_1(1-x_0)^{4k\Delta}(1-x_1)^{2k\Delta} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4k\Delta^{\theta-1}} \prod_{\lambda \geq \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2k\Delta^{2\lambda-2}}$$

$$\frac{2a\Delta^{1-\varepsilon}}{(a\Delta)^\ell} \leq y_\ell(1-x_0)^{4\ell\Delta}(1-x_1)^{2\ell\Delta} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4\ell\Delta^{\theta-1}} \prod_{\lambda \geq \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2\ell\Delta^{2\lambda-2}}$$

$$\frac{(a\Delta^{1-\varepsilon})^2}{(a\Delta)^{2m}} \leq z_m(1-x_0)^{4m\Delta}(1-x_1)^{2m\Delta} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4m\Delta^{\theta-1}} \prod_{\lambda \geq \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2m\Delta^{2\lambda-2}}$$

Let $\mathcal{P}_i$, $\mathcal{N}_i$ and x_i denote, respectively, the probabilities, number of edges and Local Lemma constants corresponding to the events of type i. By setting $\alpha = \beta = \gamma = \delta = 1000$ and $a = 4000$ and using the fact that $(1 - \frac{1}{z})^z \geq \frac{1}{4}$ $\forall z \geq 2$ we can verify that the above inequalities are satisfied, provided

$$\mathcal{P}_i \leq x_i \left(\frac{1}{4}\right)^{\mathcal{N}_i\left(\frac{2}{\alpha\Delta^\varepsilon} + \frac{1}{\beta\Delta^{2\varepsilon}} + \frac{4a}{\Delta^\varepsilon \gamma^{\lceil \frac{g}{2} \rceil - 1}(\gamma-1)} + \frac{a^2}{\Delta^{2\varepsilon}\delta^{\lceil \frac{g}{2} \rceil - 2}(\delta^2-1)}\right)}, \ \forall i \tag{2}$$

By choosing c_1 suitably large, we can verify that each of the inequalities (2) are satisfied. As a result, the inequalities corresponding to Local Lemma are also satisfied. Finally fixing $c_2 = a \cdot c_1$, the theorem is proved. □

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Appendix A

The proof of $a'(G) \leq 9\Delta$ given in [7] is based on applying a specialized version of Lovász Local Lemma. The random experiment used is to choose a colour for each edge independently and uniformly at random from a set of $C = a\Delta$ colours for some $a > 1$. Even if we ignore acyclicity and use only the events corresponding to pairs of edges coloured the same, we can easily verify that any proof ensuring properness of the random colouring with positive probability,by applying Lovász Local Lemma requires that $a \geq 4\,\mathrm{e}$. This establishes the incorrectness of the proof in [7].

Appendix B

In the proof of Theorems 1.1 and 1.2, we use the following simplification.

Since $f(z) = (1 - \frac{1}{z})^z$ and $(1 - \frac{1}{z})^z \uparrow \frac{1}{\mathrm{e}}$ we can set,

$$\Lambda = \min\left\{ f({x_0}^{-1}),\ \min_{\theta \geq \lceil \frac{g}{2} \rceil} f({x_\theta}^{-1}),\ \min_{\lambda \geq g} f({y_\lambda}^{-1}) \right\} \quad \text{and then we have,}$$

$$(1 - x_0)^{\frac{2\Delta^\eta}{\eta!}} = \left(\left(1 - \frac{1}{(\alpha\Delta)^\eta} \right)^{(\alpha\Delta)^\eta} \right)^{\frac{2}{\eta!\alpha^\eta}} \geq \Lambda^{\frac{2}{\eta!\alpha^\eta}} \quad \text{and}$$

$$\prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1 - x_\theta)^{\Delta^{2\theta-2}} \geq \Lambda^{S_1} \quad \text{and} \prod_{\lambda \geq g} (1 - y_\lambda)^{\Delta^{\lambda-2}} \geq \Lambda^{S_2}$$

$$\text{where} \quad S_1 = \sum_{\theta \geq \lceil \frac{g}{2} \rceil} \frac{1}{\beta^{2\theta-2}} \leq \frac{1}{(\beta^2 - 1)\beta^{2\lceil \frac{g}{2} \rceil - 4}} \quad \text{and}$$

$$S_2 = \sum_{\lambda \geq g} \frac{1}{\Delta\gamma^{\lambda-1}} \leq \frac{1}{\Delta\gamma^{g-2}(\gamma - 1)}.$$

Now taking roots on both sides and simplifying, $\forall k \geq \lceil \frac{g}{2} \rceil$, $\ell \geq g$ we have,

$$\frac{1}{c} \leq \frac{1}{\alpha}\Lambda^{\frac{\eta+1}{\eta}\Upsilon}, \qquad \frac{1}{c} \leq \frac{1}{\beta}\Lambda^{\frac{2k}{2k-2}\Upsilon} \quad \text{and} \quad \frac{1}{c} \leq \frac{1}{\gamma}\Lambda^{\frac{\ell}{\ell-1}\Upsilon}$$

$$\text{where} \quad \Upsilon = \frac{2}{\eta!\alpha^\eta} + \frac{1}{(\beta^2 - 1)\beta^{2\lceil \frac{g}{2} \rceil - 4}} + \frac{1}{\Delta\gamma^{g-2}(\gamma - 1)}.$$