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About acyclic edge colourings of planar graphs

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ABSTRACT

Let $G = (V, E)$ be any finite simple graph. A mapping $C : E \to [k]$ is called an *acyclic edge k-colouring* of G, if any two adjacent edges have different colours and there are no bichromatic cycles in G. In other words, for every pair of distinct colours i and j, the subgraph induced by all the edges which have either colour i or j is acyclic. The smallest number k of colours, such that G has an acyclic edge k-colouring is called the *acyclic chromatic index* of G and is denoted by $\chi'_a(G)$.

In 1991, Alon et al. [N. Alon, C.J.H. McDiarmid, B.A. Reed, Acyclic coloring of graphs, Random Structures and Algorithms 2 (1991) 277–288] proved that $\chi'_a(G) \leqslant 64\Delta(G)$ for any graph G of maximum degree $\Delta(G)$. This bound was later improved to $16\Delta(G)$ by Molloy and Reed in [M. Molloy, B. Reed, Further algorithmic aspects of the local lemma, in: Proceedings of the 30th Annual ACM Symposium on Theory of Computing, 1998, pp. 524–529].

In this paper we prove that $\chi'_a(G) \leqslant \Delta(G) + 6$ for a planar graph G without cycles of length three and that the same holds if G has an edge-partition into two forests. We also show that $\chi'_a(G) \leqslant 2\Delta(G) + 29$ if G is planar.

1. Introduction

All graphs which we consider are finite and simple. For any graph G, we denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For a vertex v, its *degree* is denoted by $d_G(v)$ or simply $d(v)$ when no confusion can arise.

For undefined concepts we refer the reader to [7].

As usual $[k]$ stands for the set $\{1, \dots, k\}$.

A mapping $C : E(G) \to [k]$ is called a *proper edge k-colouring* of a graph G provided any two adjacent edges receive different colours. A proper edge k-colouring C of G is called an *acyclic edge k-colouring* of G if there are no bichromatic cycles in G under the colouring C. In other words, for every pair of distinct colours i and j, the subgraph induced in G by all the edges which have either colour i or j, is acyclic. The smallest number k of colours such that G has an acyclic edge k-colouring is called the *acyclic chromatic index* of G and is denoted by $\chi'_a(G)$ or sometimes $a'(G)$. In this paper we follow the former notation.

It is easy to see that $\chi'_a(G) \geqslant \chi'(G) \geqslant \Delta(G)$ for any graph G, where $\chi'(G)$ stands here for the *chromatic index* of G, defined as the minimum number k such that G has a proper edge k-colouring.

The notion of acyclic colouring of a graph was introduced by Grűnbaum [9] for vertex colouring and it was later extended to the edge version as well. Fiamčík proved in [8] that $\Delta(G) \cdot (\Delta(G) - 1) + 1$ is an upper bound for the acyclic chromatic index of a graph G and conjectured that $\chi'_a(G) \leqslant \Delta(G) + 2$. In [1] Alon et al. presented a linear upper bound on $\chi'_a(G)$. They proved that $\chi'_a(G) \leqslant 64\Delta(G)$, which was later improved to $16\Delta(G)$ by Molloy and Reed [10].

In [2] Alon et al. independently made the Acyclic Edge Colouring Conjecture (the **AECC** for short), which states that $\chi'_a(G) \leqslant \Delta(G) + 2$ for all graphs G. In [2] this conjecture was proved to be true for random d-regular graphs (asymptotically) and for graphs having large girth.

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The **AECC** was also verified for some special classes of graphs including subcubic graphs [8,13], outerplanar graphs [11] and grid-like graphs [12]. For the last two classes of graphs the better bound $\Delta(G)+1$ was obtained. In [13] Skulrattanakulchai presented a linear algorithm to acyclically colour, with 5 colours, the edges of any cubic graph. Very recently Basavaraju and Chandran proved in [4], that $\chi'_a(G) \leqslant 4$ for any non-regular connected subcubic graph G. They also showed in [5] that $\chi'_a(G) \leqslant 7$ for any graph G of maximum degree 4 and in [6] that $\chi'_a(G) \leqslant \Delta(G)+1$ for any 2-degenerated graph G.

The problem of determining the acyclic chromatic index of a graph is NP-complete, what was shown by Alon and Zaks in [3]. The authors also presented a polynomial algorithm to acyclically colour the edges of a graph G with $\Delta(G)+2$ colours, provided the length of the shortest cycle in G is greater than $c(\Delta(G))^3$, for some constant c.

In this paper we provide new upper bounds for the acyclic chromatic index for the classes of planar graphs, 3-fold graphs, triangle-free planar graphs and 2-fold graphs.

A graph G is called *k-fold* if the edges of G can be partitioned into k forests.

2. Results

For brevity, throughout this section we simply use a *colouring* for an *edge colouring*. We start by introducing some necessary notations.

Let G be a graph and $v \in V(G)$. We call v a k-vertex if $d_G(v)=k$. Similarly, v is called a k^-- or a k^+-vertex if its degree is at most k or at least k, respectively. We denote by $l_k(v)$ (respectively, $l_{k^-}(v)$, $l_{k^+}(v)$) the cardinality of the set of those neighbours of v, which are k-vertices (respectively, k^--vertices, k^+-vertices).

Lemma 1. *Let G be a graph such that $|E(G)| \leqslant 2|V(G)|-1$ and $\delta(G) \geqslant 2$. Then G contains at least one of the following configurations*:

($\mathcal{A}1$) *a 2-vertex adjacent to a 5^--vertex,*
($\mathcal{A}2$) *a 3-vertex adjacent to at least two 5^--vertices,*
($\mathcal{A}3$) *a 6-vertex adjacent to at least five 3^--vertices,*
($\mathcal{A}4$) *a 7-vertex adjacent to seven 3^--vertices or*
($\mathcal{A}5$) *a vertex x such that at least $d(x)-3$ of its neighbours are 3^--vertices, and moreover one of them is of degree 2.*

Proof. We use the discharging method to prove the lemma. Let $G=(V,E)$, $\delta(G) \geqslant 2$ and $|V|=n$, $|E|=m$. We have $m \leqslant 2n-1$. Initially, we will define a mapping f on the set of vertices of G as follows: for each $x \in V$ let $f(x)=d(x)-4$. It is easy to observe that

$$\sum_{x\in V} f(x) \leqslant -2,$$

which follows from the inequality $m \leqslant 2n-1$. In the discharging step, we redistribute the values of f between adjacent vertices according to the two rules described below to obtain the function f'.

- If x is a 5^--vertex, then x does not give anything to its neighbours.
- If x is a 6^+-vertex, then x gives 1 to each 2-vertex in its neighbourhood and $\frac{1}{2}$ to each 3-vertex in its neighbourhood.

After this procedure, each vertex x has a new value $f'(x)$. But since the charges were only re-distributed, the sums of values of the functions f' and f, counting over all the vertices remain the same.

We now show that if G does not contain any of the configurations ($\mathcal{A}1$)–($\mathcal{A}5$), then for each vertex x, the value $f'(x)$ will be nonnegative. This leads us to a contradiction with the fact that

$$\sum_{x\in V} f'(x) = \sum_{x\in V} f(x) \leqslant -2. \tag{1}$$

To calculate the values $f'(x)$, we consider a number of cases, depending on $d(x)$.

- If $d(x)=2$, then $f'(x)=-2+1\cdot l_{6^+}(x)=-2+2=0$, because ($\mathcal{A}1$) does not hold.
- If $d(x)=3$, then $f'(x)=-1+\frac{1}{2}\cdot l_{6^+}(x) \geqslant -1+\frac{1}{2}\cdot 2=0$, because ($\mathcal{A}2$) does not hold.
- If $d(x)=4$, then $f'(x)=f(x)=0$.
- If $d(x)=5$, then $f'(x)=f(x)=1$.
- If $d(x)=6$, then $f'(x)=2-1\cdot l_2(x)-\frac{1}{2}\cdot l_3(x)$. Let us initially assume that x has at least one neighbour of degree 2. Now, since ($\mathcal{A}5$) does not occur, x can have at most $d(x)-4=6-4=2$ neighbours of degree at most 3, and hence $f'(x) \geqslant 2-2=0$. For the case when x does not have any neighbour of degree 2, one can observe that $f'(x)=2-\frac{1}{2}\cdot l_3(x) \geqslant 2-\frac{1}{2}\cdot 4=0$, because ($\mathcal{A}3$) does not hold.
- If $d(x)=7$, then $f'(x)=3-1\cdot l_2(x)-\frac{1}{2}\cdot l_3(x)$. First we will assume that x has at least one neighbour of degree 2. Then because ($\mathcal{A}5$) does not occur, we notice that x can have at most $d(x)-4=7-4=3$ neighbours of degree at most 3 and therefore $f'(x) \geqslant 3-3=0$. In the other case where x does not have a degree 2 neighbour, we have $f'(x)=3-\frac{1}{2}\cdot l_3(v) \geqslant 3-\frac{1}{2}\cdot 6=0$, since ($\mathcal{A}4$) does not hold.
- If $d(x) \geqslant 8$, then if x has at least one neighbour of degree 2 and ($\mathcal{A}5$) does not hold, x can have at most $d(x)-4$ neighbours of degree at most 3 and therefore $f'(x) \geqslant 0$. For the case when x does not have a neighbour of degree 2, we have $f'(x)=d(x)-4-\frac{1}{2}\cdot l_3(x) \geqslant d(x)-4-\frac{1}{2}\cdot d(x)=\frac{1}{2}\cdot d(x)-4 \geqslant 0$.

Since for each vertex x of G the value $f'(x)$ is nonnegative, we obtain a contradiction with the inequality (1). □

In the sequel we frequently use the following notations.

Let C be an acyclic k-colouring of a graph G. For any vertex v, we denote by $C(v)$ the set of colours assigned by C to the edges incident to v. For $W \subseteq V(G)$ we define $C(W)=\bigcup_{w\in W} C(w)$. For an edge uv, $C(uv)$ is the colour of uv in C.

If v and u are two distinct vertices of G then let $W_G(v,u)$ stand for the set of neighbours w of the vertex v in G for which $C(vw) \in C(u)$. Notice that the order of v and u is important here and that the set $W_G(v,u)$ could be empty.

The next lemma is a modification of the Extension Lemma stated in [11].

Lemma 2. *Let G be a graph, $uv \in E(G)$ and let C be an acyclic k-colouring of $G - uv$. If $|C(v) \cup C(u) \cup C(W_{G-vu}(v,u))| < k$, then the colouring C can be extended to an acyclic k-colouring of G.*

Proof. It is enough to colour the edge uv with any colour α from the set $[k] - (C(u) \cup C(v) \cup C(W_{G-vu}(v,u)))$, to obtain an acyclic k-colouring of G. $\square$

Lemma 3. *If G is a graph such that $|E(G')| \leqslant 2|V(G')| - 1$ for each $G' \subseteq G$, then $\chi'_a(G) \leqslant \Delta(G) + 6$.*

Proof. Suppose H is a counterexample to Lemma 3 with the number of edges as least as possible. Let $k = \Delta(H) + 6$.

We may assume without loss of generality that H is 2-connected. Otherwise we can obtain an acyclic k-colouring of each its 2-connected component and combine them (by renaming some colours if needed) to get an acyclic k-colouring of the entire graph.

Hence we have $\delta(H) \geqslant 2$ and, by Lemma 1, the graph H contains at least one of the configurations $(\mathcal{A}1)$–$(\mathcal{A}5)$.

According to Lemma 2, it is sufficient to show that there exists an edge vu and an acyclic k-colouring C of $H - vu$ such that $|C(v) \cup C(u) \cup C(W_{H-vu}(v,u))| < k$. We will consider a number of cases, depending on which of the configurations $(\mathcal{A}1)$–$(\mathcal{A}5)$ occurs in H. In each case we will point out such an edge which we can use with Lemma 2 to obtain a contradiction.

Configuration $(\mathcal{A}1)$. If H contains a 2-vertex x adjacent to a 5^--vertex y, then let z be the remaining neighbour of x. Moreover, let $H' = H - xz$. Clearly, $\chi'_a(H') \leqslant k$. Let C be any acyclic k-colouring of H' and assume that $C(x) = a$. We consider two cases.

Case 1. If $|C(z) \cap \{a\}| = 0$ then $|C(z) \cup \{a\}| \leqslant \Delta(H)$ and $W_{H'}(x,z) = \emptyset$. Therefore, from Lemma 2, H has an acyclic k-colouring, a contradiction.

Case 2. If $|C(z) \cap \{a\}| = 1$ then $W_{H'}(x,z) = \{y\}$. From the fact $d_H(y) \leqslant 5$ we have $|C(y)| \leqslant 5$, therefore $|C(x) \cup C(z) \cup C(y)| \leqslant \Delta(H) + 3$. By Lemma 2, we can extend the colouring C to an acyclic k-colouring of H, a contradiction.

Configuration $(\mathcal{A}2)$. If there is a 3-vertex x in H adjacent to two 5^--vertices z and z_1, then let y be the third neighbour of x. Moreover let $H' = H - xz$. Since H' has less edges than H, $\chi'_a(H') \leqslant k$. Let C be an acyclic k-colouring of H'. We consider the following cases.

Case 1. $|C(z) \cap C(x)| \leqslant 1$. We have

$$\left|C(x) \cup C(z) \cup C\big(W_{H'}(x,z)\big)\right| \leqslant \Delta(H) + 4$$

and, by Lemma 2, we can extend the colouring C to an acyclic k-colouring of H, a contradiction.

Case 2. $|C(z) \cap C(x)| = 2$.

Subcase 2.1. If $C(xy) \notin C(z_1)$, then $|C(z) \cup C(z_1) \cup C(x)| \leqslant \Delta(H) + 3$. Therefore we can recolour (in H') the edge xz_1 with a colour $\alpha \notin C(z) \cup C(z_1) \cup C(x)$, obtaining an acyclic k-colouring C' of H' reducing it to the previous case.

Subcase 2.2. $C(xy) \in C(z_1)$.

Case 2.2(a). If $C(xz_1) \notin C(y)$, then $|C(x) \cup C(y) \cup C(z)| \leqslant \Delta(H) + 3$. Thus we can recolour, in H', the edge xy with a colour $\alpha \notin C(x) \cup C(y) \cup C(z)$ to obtain an acyclic k-colouring C' of H' and we are in the first case.

Case 2.2(b). If $C(xz_1) \in C(y)$, then $|C(x) \cup C(z) \cup C(z_1) \cup C(y)| \leqslant \Delta(H) + 5$ and, by Lemma 1, we can extend the colouring C to an acyclic k-colouring of H, a contradiction.

Configuration $(\mathcal{A}3)$. If in H, there is a 6-vertex x adjacent to five 3^--vertices say z, z_1, z_2, z_3 and z_4, then let y be the remaining neighbour of x. Moreover let $H' = H - xz$. Since H' has less edges than H, $\chi'_a(H') \leqslant k$. Let C be an acyclic k-colouring of H'.

Case 1. $|C(x) \cap C(z)| \leqslant 1$. It follows

$$\left|C(x) \cup C(z) \cup C\big(W_{H'}(x,z)\big)\right| \leqslant \Delta(H) + 5$$

and hence, by Lemma 2, we can extend the colouring C to an acyclic k-colouring of H, a contradiction.

Case 2. $|C(x) \cap C(z)| = 2$ and $C(xy) \notin C(z)$. Clearly, $|C(x) \cup C(z) \cup C(W_{H'}(x,z))| \leqslant 9$. Therefore, according to Lemma 2 and since $\Delta(H) \geqslant 6$, H has an acyclic k-colouring, a contradiction.

Case 3. $|C(x) \cap C(z)| = 2$ and $C(xy) \in C(z)$. Assume without loss of generality that $C(xz_1) \in C(z)$.

Subcase 3.1. If $|C(y) \cap C(x)| = 1$, then we recolour the edge xy (in H') with a colour $\alpha \notin C(x) \cup C(y)$ to obtain an acyclic k-colouring C' and we are in the previous case.

Subcase 3.2. If $|C(y) \cap C(x)| \geqslant 2$, then $|C(x) \cup C(z) \cup C(W_{H'}(x,z))| \leqslant \Delta(H) + 5$ and, by Lemma 2, we can extend the colouring C to an acyclic k-colouring of H again a contradiction.

Configuration $(\mathcal{A}4)$. If H has a 7-vertex x adjacent to seven 3^--vertices, then let z be one of its neighbours and let $H' = H - xz$. Since H' has less edges than H, $\chi'_a(H') \leqslant k$. Let C be an acyclic k-colouring of H'. We can observe that $|C(x) \cup C(z) \cup C(W_{H'}(x,z))| \leqslant 10$. Therefore, according to Lemma 2 and since $\Delta(H) \geqslant 7$, H has an acyclic k-colouring, a contradiction.

Configuration $(\mathcal{A}5)$. If none of the cases $(\mathcal{A}1)$–$(\mathcal{A}4)$ occurs, then there must be a vertex x in H such that at least $d_H(x) - 3$ of its neighbours are 3^--vertices and one of them, say z, is of degree 2. Let us consider the graph $H' = H - xz$. Since H' has less edges than H, $\chi'_a(H') \leqslant k$. Let C be an acyclic k-colouring of H'. Let $C(z) = \{a\}$, $C_1 = \{C(xy)\colon\ y \in N_H(x) \text{ and } d_H(y) > 3\}$, and $C_2 = \{C(xy)\colon\ y \in N_H(x) - \{z\} \text{ and } d_H(y) \leqslant 3\}$.

Case 1. If $a \notin C_1$, then $|C(x) \cup C(z) \cup C(W_{H'}(x,z))| \leqslant \Delta(H) + 1$. Therefore, by Lemma 2, H has an acyclic k-colouring, a contradiction.

Case 2. On the other hand, if $a \in C_1$, then let y be the neighbour of z in H'. There is a colour $\alpha \notin C_1 \cup C(y)$ such that we can recolour (in the graph H') the edge zy with α, obtaining in this way an acyclic k-colouring C' of H' and we will fall in the previous case. □

We can now formulate two theorems, which provide upper bound on the acyclic chromatic index for 2-fold graphs and the class of planar graphs without cycles of length three.

Theorem 4. *Let G be any 2-fold graph. Then*

$$\chi'_a(G) \leqslant \Delta(G) + 6.$$

Proof. It follows from Lemma 3 and the two known facts:

(i) any subgraph of a 2-fold graph is also 2-fold;
(ii) in any 2-fold graph G, of order at least 2, $|E(G)| \leqslant 2|V(G)| - 2$. □

Theorem 5. *Let G be any planar graph without cycles of length three. Then*

$$\chi'_a(G) \leqslant \Delta(G) + 6.$$

Proof. It is known that, $|E(G)| \leqslant 2|V(G)| - 4$ for any planar graph G without cycles of length three having order at least 3. This can be proved using Euler's formula. Clearly, this property holds for any subgraph also. Therefore, from Lemma 3, $\chi'_a(G) \leqslant \Delta(G) + 6$. □

Lemma 6. *Let G be any graph such that $|E(G)| \leqslant 3|V(G)| - 1$ and $\delta(G) \geqslant 3$. Then at least one of the following configurations occurs in G:*

($\mathcal{B}1$) *a 3-vertex adjacent to an 11^--vertex,*
($\mathcal{B}2$) *a 4-vertex adjacent to at least two 11^--vertices,*
($\mathcal{B}3$) *a 5-vertex adjacent to at least three 11^--vertices,*
($\mathcal{B}4$) *a vertex x such that $12 \leqslant d(x) \leqslant 14$ and at least $d(x) - 2$ of its neighbours are 5^--vertices,*
($\mathcal{B}5$) *a 15-vertex adjacent to at least fourteen 5^--vertices,*
($\mathcal{B}6$) *a vertex x such that $16 \leqslant d(x) \leqslant 17$ and all its neighbours are 5^--vertices,*
($\mathcal{B}7$) *a vertex x such that at least $d(x) - 5$ of its neighbours are 5^--vertices and at least one of them is of degree 3.*

Proof. Let $G = (V, E)$ be such that $\delta(G) \geqslant 3$ and $m \leqslant 3n - 1$, where $|V| = n$, $|E| = m$. First, we define a function f on V as follows: for each $x \in V$ let $f(x) = d(x) - 6$. Clearly,

$$\sum_{x \in V} f(x) \leqslant -2,$$

which follows from the inequality $m \leqslant 3n - 1$. In the next step we will distribute the values of f between adjacent vertices, according to the two rules described below to obtain the function f'.

- If x is a 11^--vertex, x does not give anything to its neighbours.
- If x is a 12^+-vertex, then x gives 1 to each 3-vertex in its neighbourhood, $\frac{2}{3}$ to each 4-vertex in its neighbourhood and $\frac{1}{3}$ to each 5-vertex in its neighbourhood.

Now each vertex x has a new value $f'(x)$, but the sums of values of the functions f' and f, counting over all the vertices, remain the same.

In the following we show that, if G does not contain any of the configurations ($\mathcal{B}1$)–($\mathcal{B}7$), then $f'(x)$ will be nonnegative for each x, which will lead us to a contradiction with the fact that

$$\sum_{x \in V} f'(x) = \sum_{x \in V} f(x) \leqslant -2. \tag{2}$$

We consider a number of cases depending on the degree of x.

- If $d(x) = 3$, then $f'(x) = -3 + 1 \cdot l_{12^+}(x) = -3 + 3 = 0$, because ($\mathcal{B}1$) does not hold.
- If $d(x) = 4$, then $f'(x) = -2 + \frac{2}{3} \cdot l_{12^+}(x) \geqslant -2 + \frac{2}{3} \cdot 3 = 0$, because ($\mathcal{B}2$) does not hold.
- If $d(x) = 5$, then $f'(x) = -1 + \frac{1}{3} \cdot l_{12^+}(x) \geqslant -1 + \frac{1}{3} \cdot 3 = 0$, because ($\mathcal{B}3$) does not hold.
- If $6 \leqslant d(x) \leqslant 11$, then $f'(x) = f(x) \geqslant 0$.
- If $12 \leqslant d(x) \leqslant 14$, then $f'(x) = d(x) - 6 - 1 \cdot l_3(x) - \frac{2}{3} \cdot l_4(x) - \frac{1}{3} \cdot l_5(x)$. If we assume that x has at least one neighbour of degree 3, then since ($\mathcal{B}7$) does not occur, x can have at most $d(x) - 6$ neighbours which are 5^--vertices, and hence $f'(x) \geqslant 0$. If the case is that none of the neighbours of x is a 3-vertex, we observe that $f'(x) \geqslant d(x) - 6 - \frac{2}{3} \cdot l_{5^-}(x) \geqslant d(x) - 6 - \frac{2}{3} \cdot (d(x) - 3) \geqslant 0$, since ($\mathcal{B}4$) does not hold.
- When $d(x) = 15$, $f'(x) = 9 - l_3(x) - \frac{2}{3} \cdot l_4(x) - \frac{1}{3} \cdot l_5(x)$. If we assume that x is adjacent to a 3-vertex, then from the fact that ($\mathcal{B}7$) is absent, we can see that x can be adjacent to at most $d(x) - 6$ vertices of degree at most 5 implying $f'(x) \geqslant 0$. If x is not adjacent to a 3-vertex, then $f'(x) \geqslant 9 - \frac{2}{3} \cdot l_{5^-}(x) \geqslant 9 - \frac{2}{3} \cdot 13 \geqslant 0$, because ($\mathcal{B}5$) does not hold.
- If $16 \leqslant d(x) \leqslant 17$, then $f'(x) = d(x) - 6 - l_3(x) - \frac{2}{3} \cdot l_4(x) - \frac{1}{3} \cdot l_5(x)$. If we assume that x is adjacent to a 3-vertex, then since ($\mathcal{B}7$) does not occur we know that x can be adjacent to at most $d(x) - 6$ vertices of degree at most 5 and $f'(x) \geqslant 0$. On the other hand, if x is not adjacent to a 3-vertex then $f'(x) \geqslant d(x) - 6 - \frac{2}{3} \cdot l_{5^-}(x) \geqslant d(x) - 6 - \frac{2}{3} \cdot (d(x) - 1) \geqslant 0$, since ($\mathcal{B}6$) does not occur.
- When $d(x) \geqslant 18$, $f'(x) = d(x) - 6 - l_3(x) - \frac{2}{3} \cdot l_4(x) - \frac{1}{3} \cdot l_5(x)$. Supposing that x is adjacent to a 3-vertex and since ($\mathcal{B}7$) does not occur, x can be adjacent to at most $d(x) - 6$ vertices of degree at most 5 and $f'(x) \geqslant 0$. Again, if x is not adjacent to a 3-vertex then $f'(x) \geqslant d(x) - 6 - \frac{2}{3} \cdot l_{5^-}(x) \geqslant d(x) - 6 - d(x) \cdot \frac{2}{3} \geqslant 0$.

Since for every vertex x of G the value $f'(x)$ is nonnegative, we obtain a contradiction with the inequality (2). □

Lemma 7. *If G is a graph such that $|E(G')| \leqslant 3|V(G')| - 1$ for each $G' \subseteq G$, then*

$\chi'_a(G) \leqslant 2\Delta(G) + 29.$

Proof. As in proof of Lemma 3 above, assume that H is a minimal counterexample to Lemma 7 (with the number of edges as least as possible).

Let k stands for $2\Delta(H) + 29$.

Once again we can assume without loss of generality that H is 2-connected and hence $\delta(H) \geqslant 2$. One can observe that if $\delta(H) = 2$ then $\chi'_a(H) \leqslant k$. Clearly, if e is an edge incident to a 2-vertex, then there is an acyclic k-colouring C of $H - e$. We are using more than $2\Delta(H)$ colours, therefore this colouring can be easily extend to an acyclic k-colouring of H.

Hence $\delta(H) \geqslant 3$. Then, by Lemma 6, H contains at least one of the configurations $(\mathcal{B}1)$–$(\mathcal{B}7)$.

According to Lemma 2, it is enough to show that $|C(v) \cup C(u) \cup C(W_{H-vu}(v,u))| < k$ for some edge vu and an acyclic k-colouring C of $H - vu$. We will consider different cases depending on $\delta(H)$ and which of the configurations $(\mathcal{B}1)$–$(\mathcal{B}7)$ occurs in H. As in proof of Lemma 3 we will try to find a suitable edge in order to use Lemma 2 and make a contradiction with the fact that H is a minimal counterexample.

Configuration $(\mathcal{B}1)$. If there is a 3-vertex x, adjacent to a 11^--vertex y in H, then assume that z is another neighbour of x (in H) and let $H' = H - xz$. From the fact that H is a minimal counterexample we see that H' has an acyclic k-colouring, say C. Moreover, since $d_H(y) \leqslant 11$, we have $|C(z) \cup C(x) \cup C(W_H(x,z))| \leqslant 2\Delta(H) + 8$. According to Lemma 2, it follows that H has an acyclic k-colouring, a contradiction.

Configuration $(\mathcal{B}2)$. If H contains a 4-vertex x adjacent to at least two 11^--vertices say y_1, y_2, let z be any other neighbour of x (in H) and let $H' = H - xz$. Since H is a minimal counterexample, H' has an acyclic k-colouring. Moreover, because $d_H(y_1), d_H(y_2) \leqslant 11$, we have $|C(z) \cup C(x) \cup C(W_{H'}(x,z))| \leqslant 2\Delta(H) + 18$. From Lemma 2 it follows that H has an acyclic k-colouring, a contradiction.

Configuration $(\mathcal{B}3)$. If in H there is a 5-vertex x adjacent to at least three 11^--vertices y_1, y_2, y_3, then let z be any other neighbour of x. Let $H' = H - xz$. As before, H' has an acyclic k-colouring C. We also have $d_H(y_1)$, $d_H(y_2), d_H(y_3) \leqslant 11$ implying $|C(z) \cup C(x) \cup C(W_{H'}(x,z))| \leqslant \max\{\Delta(H) + 30, 2\Delta(H) + 28\} \leqslant 2\Delta(H) + 28$, because $\Delta(H) \geqslant 5$. As in the previous case, it follows that H has an acyclic k-colouring, a contradiction.

Configuration $(\mathcal{B}4)$. If H has a vertex x of degree 12 or 13 or 14, adjacent to at least $d_H(x) - 2$ vertices of degree at most 5 then let z be any of them and let $H' = H - xz$. From the fact that H is *a* minimal counterexample we have that H' has an acyclic k-colouring C. From the fact that all, except at most two, neighbours of x in H are of degree at most 5 we have $|C(z) \cup C(x) \cup C(W_{H'}(x,z))| \leqslant 2\Delta(H) + 19$. According to Lemma 2, it follows that H has an acyclic k-colouring, a contradiction.

Configuration $(\mathcal{B}5)$. If there is a 15-vertex x adjacent to at least fourteen 5^--vertices, then let z be any of them and let $H' = H - xz$. As before H' has an acyclic k-colouring C. We notice that $|C(z) \cup C(x) \cup C(W_{H'}(x,z))| \leqslant \Delta(H) + 25 \leqslant 2\Delta(H) + 10$, because $\Delta(H) \geqslant 15$. It follows from Lemma 2 that H has an acyclic k-colouring, a contradiction.

Configuration $(\mathcal{B}6)$. If H contains a vertex x of degree either 16 or 17, such that all its neighbours are 5^--vertices, then let z be any of this neighbours and let $H' = H - xz$. From the fact that H is a minimal counterexample we have that H' has an acyclic k-colouring C. Moreover, since all neighbours of x in H are of degree at most 5 we have $|C(z) \cup C(x) \cup C(W_{H'}(x,z))| \leqslant 32 \leqslant 2\Delta(H)$. According to Lemma 2, it follows that H has an acyclic k-colouring, a contradiction.

Configuration $(\mathcal{B}7)$. If H has a vertex x, having at least $d_H(x) - 5$ neighbours of degree at most 5, one of which say z, is of degree 3, then let $H' = H - xz$. From the fact that H is a minimal counterexample, H' has an acyclic k-colouring C. Let $C_1 = \{C(xy)\colon\ y \in N_H(x) \text{ and } d_H(y) > 5\}$, $C_2 = \{C(xy)\colon\ y \in N_H(x),\ y \neq z \text{ and } d_H(y) \leqslant 5\}$.

If $C(z) \cap C_1 = \emptyset$, then $|C(z) \cup C_1 \cup C_2 \cup C(W_{H'}(x,z))| \leqslant \Delta(H) + 7$. From Lemma 2 it follows that H has an acyclic k-colouring, a contradiction.

If $C(z) \cap C_1 \neq \emptyset$, then let z_1, z_2 be neighbours of z in H'. If $C(zz_1) \in C_1$, then there is a colour $\alpha \notin C_1 \cup C(z_1) \cup C(z_2) \cup C(z)$, with which we can recolour the edge zz_1 with α, to obtain an acyclic k-colouring C' of H'. It follows from the fact that $|C_1 \cup C(z_1) \cup C(z_2) \cup C(z)| \leqslant 2\Delta(H) + 4$. Further, if $C(zz_2) \in C_1$, then there is a colour $\beta \notin C_1 \cup C(z_1) \cup C(z_2) \cup C(z) \cup \{\alpha\}$, which can be used to recolour the edge zz_2 to obtain an acyclic k-colouring C'' of H'. In each situation we are back in the previous case. □

We are now able to generalise the bound given in Theorem 5 to the classes of planar graphs and 3-fold graphs. The result is not tight, but it is the best known upper bound.

Theorem 8. *If G is planar, then $\chi'_a(G) \leqslant 2\Delta(G) + 29$.*

Proof. It is a known fact that for any planar graph G, $|E(G)| \leqslant 3|V(G)| - 6$. Note that this property also holds for any subgraph. Therefore we can apply Lemma 7 to get the required result. □

Theorem 9. *If G is 3-fold, then $\chi'_a(G) \leqslant 2\Delta(G) + 29$.*

Proof. If G is 3-fold, then G is the union of three forests and thus $|E(G)| \leqslant 3|V(G)| - 3$. Obviously, this property also holds for any subgraph. Therefore, by Lemma 7, $\chi'_a(G) \leqslant 2\Delta(G) + 29$. □

3. Concluding remarks

Here we have made use of the discharging method to get some local structure for graphs with bounded number of edges like planar graphs and 2-fold graphs. This was further used to improve the bounds for the acyclic chromatic index. It might be interesting to look at other classes of graphs where the number of edges is linear. It is also interesting to see if the gap between lower and upper bound for planar graphs can be reduced.

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