Strong Chromatic Index of 2-Degenerate Graphs

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Abstract: We prove that the strong chromatic index of a 2-degenerate graph is linear in the maximum degree Δ. This includes the class of all chordless graphs (graphs in which every cycle is induced) which in turn includes graphs where the cycle lengths are multiples of four, and settles a problem by Faudree et al. (Ars Combin 29(B) (1990), 205–211). J. Graph Theory 73: 119–126, 2013

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1. INTRODUCTION

All graphs we consider are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$, respectively, denote the sets of vertices and edges of G. A *proper edge k-coloring* of a graph G is a map $\mathcal{C} : E(G) \mapsto [k]$ so that adjacent edges (edges sharing a common vertex) receive different colors (numbers), where $[k] = \{1, 2, \dots, k\}$. The smallest positive integer k for which a graph G admits a proper edge k-coloring is known as the *chromatic index* of G denoted $\chi'(G)$. An analogous notion for the vertex coloring is known as the *chromatic number*.

A proper edge coloring is called a *strong edge coloring*, if every color class is an induced matching in G. In other words, for any edge $e = uv$, the sets of colors *seen* by u and v has exactly one color in common. (In an edge coloring, we say that a vertex *sees* color c, if c is assigned to any of the edges incident to it). That is, the distance between any two edges having the same color is at least two. The minimum k such that G admits a strong edge k-coloring is the *strong chromatic index* of G denoted $\chi'_s(G)$. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in G.

Molloy and Reed [10] proved a conjecture by Erdős and Nešetřil (see [5]) that for large Δ, there is a positive constant c such that $\chi'_s(G) \leq (2 - c)\Delta^2$. Mahdian [9] proved that for a C_4-free graph G, $\chi'_s(G) \leq (2 + o(1))\Delta^2 / \ln \Delta$.

For integers $0 \leq \ell \leq k \leq m$, $S_m(k, \ell)$ is the bipartite graph with vertex set $\{x \subseteq [m] : |x| = k \text{ or } \ell\}$ and a k-subset x is adjacent to an ℓ-subset y if $y \subseteq x$. Quinn and Benjamin [14] proved that $S_m(k, \ell)$ has strong chromatic index $\binom{m}{k-\ell}$. The Θ-*graph* $\Theta(G)$ of a partial cube (distance-invariant subgraph of some n-cube) G is the intersection graph of the equivalence classes of the *Djoković-Winkler relation* Θ defined on the edges of G such that xy and uv are in relation Θ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. Recently, Šumenjak [15] showed that the strong chromatic index of a tree-like partial cube graph G is at most the chromatic number of $\Theta(G)$.

Some of the many unsolved conjectures include $\chi'_s(G) \leq 5\Delta^2/4$ for all graphs, $\chi'_s(G) \leq \Delta^2$ for bipartite graphs, and $\chi'_s(G) \leq 9$ for 3-regular planar graphs. See the open problems pages of Douglas West for more details.

A *chord* in a graph is an edge that joins two nonconsecutive vertices of a cycle. The set of all graphs in which every cycle is induced, and so do not contain chords for any cycle, are called *chordless* graphs [7, 11]. A 2-connected graph is *minimally* 2*-connected (block line critical)*, if for any $e \in E$, $G - e$ is not 2-connected. It can be easily verified that a graph is minimally 2-connected if and only if it is 2-connected and chordless.

Faudree et al. [6] proved that for graphs where all cycle lengths are multiples of four, $\chi'_s(G) \leq \Delta^2$. They mention that this result probably could be improved to a linear function of the maximum degree. Brualdi and Quinn [3] improved the upper bound to $\chi'_s(G) \leq \alpha\beta$ for such graphs, where α and β are the maximum degrees of the respective partitions. Nakprasit [12] proved that if G is bipartite and the maximum degree of one partite set is at most 2, then $\chi'_s(G) \leq 2\Delta$.

A graph is 2*-degenerate*, if every subgraph has a vertex of degree at most 2. For any minimally 2-connected graph G, $G \setminus D$ is a forest (possibly empty), where D is the set of all vertices of degree 2 [1]. This implies that they are 2-degenerate. In this work, we prove that $\chi'_s(G)$ is linear in Δ for a 2-degenerate graph G. Notice that graphs whose cycles have lengths divisible by 4 are chordless graphs, for otherwise a chord will divide a cycle of length $4k$ into two cycles with sum of lengths $4k + 2$ and then at least one

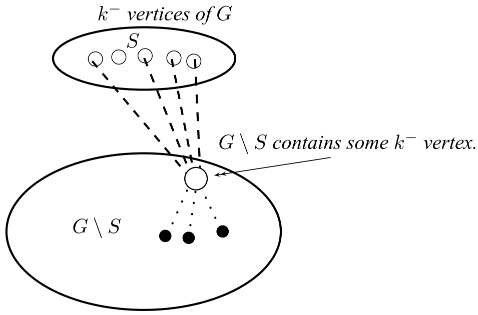

FIGURE 1. Vertex v in a k-degenerate graph.

of the cycle has a length not divisible by 4. Consequently, our result settles the above question by Faudree et al. in the positive.

Specifically, we prove the following.

Theorem 1. *If G is* 2*-degenerate, then* $\chi'_s(G) \leq 10\Delta - 10$.

The class of chordless graphs is a strict subclass of 2-degenerate graphs and has more desirable structural properties that enable us to improve the upper bound as follows.

Theorem 2. *If G is chordless, then* $\chi'_s(G) \leq 8\Delta - 6$.

In the following, the degree of a vertex v is denoted $d(v)$. A vertex of degree k is called a *k-vertex*. Vertices of degree at most k and at least k are, respectively, called k^-*-vertex* and k^+*-vertex*. For a graph G and a vertex set $S \subseteq V(G)$, let $G \setminus S$ denote the subgraph induced on $V(G) \setminus S$. For an edge e, let $G - e$ be the subgraph by deleting e from $E(G)$. An edge incident to a 1-vertex is called a *pendant edge*.

2. PROOFS

In this section, we prove the results. First, we consider 2-degenerate graphs and then show how the proof can be modified to improve the bounds for chordless graphs.

A. 2-Degenerate Graphs

In this subsection, we prove our main result for 2-degenerate graphs. The following simple lemma throws light into the nice structure of k-degenerate graphs that enables us to apply the coloring procedure.

Lemma 3. *If G is a k-degenerate graph, then there is some* $v \in V(G)$ *such that at least* $\max\{1, d(v) - k\}$ *of its neighbors are* k^-*-vertices.*

Proof. By definition, G contains k^--vertices. Let S be the set of k^--vertices of G. Again, by definition, $G \setminus S$ (unless trivial) contains k^--vertices. Let v be any k^--vertex in $G \setminus S$. It can have at most k neighbors in $G \setminus S$ and all the remaining $d(v) - k$ neighbors should be in S. By selection, v has at least one neighbor in S. See Figure 1 for illustration. ■

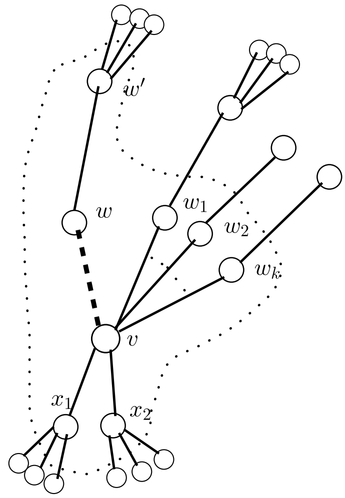

FIGURE 2. The local structure of a 2-degenerate graph.

For a 2-degenerate graph, this implies the following fact.

Fact 4. *We can construct any* 2*-degenerate graph from the trivial graph by adding edges ab such that at most* 2 *neighbors of a have degree more than* 2*, and degree of b is at most* 2 *(in the present graph).*

Proof of Theorem 1. We use an induction on the number of edges. Let $\mathcal{B} = \{1, 2, \ldots, 5\Delta - 5\}$ and $\mathcal{A} = \{1', 2', \ldots, (5\Delta - 5)'\}$ be the *base set* and *additional set* of colors, respectively. For a set $X \subseteq \mathcal{B} \cup \mathcal{A}$, the set $\mathcal{B}(X)$ is the set obtained from X by ignoring the apostrophe. For example, $\mathcal{B}(\{a, b, c, b', d'\}) = \{a, b, c, d\}$. We use the following induction hypothesis. ■

For any 2*-degenerate graph G on at most m edges, there is a strong edge coloring* $\mathcal{C}$ *that satisfies the following.*

(1) *The coloring* $\mathcal{C}$ *uses at most* $10\Delta - 10$ *colors.*
(2) *Every pendant edge (if any) is colored from the base set* $\mathcal{B}$*.*
(3) *For a pendant edge e colored c, no edge at a distance* 1 *is colored* c'*.*
(4) *No pair of edges incident to a vertex is coloured c and* c'*.*

Base cases can be easily verified. Let G be a 2-degenerate graph on $m + 1$ edges. We may assume that $\Delta > 2$ as otherwise the bounds are trivial.

By Lemma 3, G contains a vertex v such that at least $d(v) - 2$ of its neighbors are 2^--vertices, see Figure 2 for illustration. Therefore, at most 2 neighbors of v can be of higher degree. Let w be a 2^--neighbor of v. By assumption, the graph $H = G - vw$ has a strong edge coloring that satisfies the induction hypothesis.

Now consider G. When we add the edge vw, the distance between the edges incident with v and the unique other edge incident with w becomes 1. This may create some discrepancy to the existing strong edge coloring. We first show how to color the edge vw and then describe how to deal with the discrepancy.

Consider the set X of all colors used by the edges incident to the neighbors of v and w, the edges enclosed by the dotted region in Figure 2. The total number of such edges is at most $3 \times \Delta + (d(v) - 3) \times 2 < 5\Delta - 5$. It is easy to see that $|\mathcal{B}(X)| \leq |X| < 5\Delta - 5$. This is clearly an upper bound on the number of colors unavailable to color vw. Since the *base set* $\mathcal{B}$ has $5\Delta - 5$ colors, we can always choose one color for the edge vw from the set $\mathcal{B} \setminus \mathcal{B}(X)$ of base colors.

Since the degree of w is at most 2, it can have at most one other neighbor, say w'. If there is no such vertex, the coloring satisfies the induction hypothesis and we are done. If it exists, the edge ww' may have received the same color as one of the edges incident to v, say vv'. If $d(w') \leq 3$, at most 3Δ colors are unavailable for ww', and we can find a new color from the *base set* $\mathcal{B}$ to recolor it to get a strong edge coloring of G that satisfies the hypothesis. Similarly, if v' is any of the 2^- neighbors of v or $d(v') \leq 3$, then we can recolor vv' to get a strong edge coloring of G.

The only remaining case is that $d(w') > 3$ and $d(v') > 3$. In this case, if the edge ww' was colored c, we recolor it with the corresponding color c' from the *additional set* $\mathcal{A}$. We call this process as *priming* the color of an edge.

We claim that the resultant coloring satisfies the induction hypothesis.

Proof. It is easy to see that the coloring satisfies Property 1 of the hypothesis because it uses at most $10\Delta - 10$ colors. Property 2 is also satisfied because, we never *prime* a pendant edge and always color a new edge from the *base set* $\mathcal{B}$.

Observe that in $G - e$, the edge ww' is a pendant edge and satisfies Property 3. Since the coloring of $G - e$ is a strong edge coloring, no edge at a distance 1 from ww' was colored c. Similarly, no edge at a distance 1 is colored c' due to Property 3. In G, the edge vv' that is colored c is not a pendant edge, as $d(v), d(v') > 3$. The edge ww' is not a pendant edge in G, and thus after recoloring, Property 3 also holds. Finally, Property 4 is unaffected by the recolouring. Therefore, we see that the coloring of G is a strong edge coloring since the only conflicting edge has been recolored satisfying all the constraints. Note that vertex v does not see colour c' due to Property 4.

Thus, we have extended the strong edge coloring of $H = G - e$ to G. Hence, using induction and Fact 4, the theorem follows. ■

B. Chordless Graphs

Next, we prove Theorem 2, which improves this bound for chordless graphs.

Proof of Theorem 2. We know that every minimally 2-connected graph is 2-degenerate with minimum degree 2 [1]. Let G be a minimally 2-connected graph and let S be the set of vertices of degree 2 in G. We call a path $P = (v_1, v_2, \ldots, v_t)$, $t \geq 3$, in G an S-path, if each of its intermediate vertices $v_2, v_3, \ldots, v_{t-1}$ are of degree 2. We have the following result due to Plummer [13] and independently due to Dirac [4].

Theorem 5 (Dirac and Bollobás [1, 4]). *If G is a minimally 2-connected graph, then for any edge e, $G - e$ decomposes into blocks such that each block is either an edge or a minimally 2-connected subgraph of G.*

Theorem 6 (Plummer [13]). *If G is a minimally 2-connected graph that is not a cycle and S is the set of all degree 2 vertices of G, then $G - S$ is a forest with components*

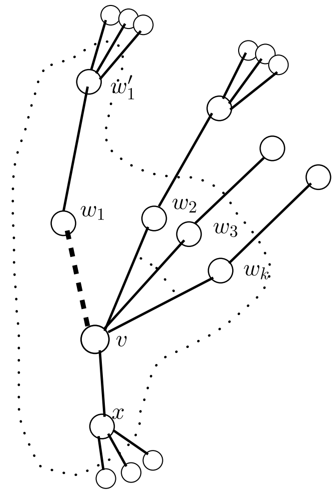

FIGURE 3. The local structure of a minimally 2-connected graph.

$T_1, T_2, \ldots, T_s$ with $s \geq 2$ such that there is no S-path joining two vertices of the same tree T_i.

The following lemma is immediate from the above theorems.

Lemma 7. *Every chordless graph G contains some vertex x such that at least $d(x) - 1$ of its neighbors are 2^--vertices.*

Proof. If G is a cycle or a path, the lemma is trivial. Otherwise by Theorem 5, each block is either an edge or minimally 2-connected. Therefore, by Theorem 6, we have that $G - S$ is a forest with at least two components. Since each component is a tree, it contains some vertex of degree at most 1. Let x be any such vertex. Observe that each neighbor of x other than its unique neighbor (if any) in its component belongs to S and thus has degree at most 2 in G. The lemma follows. See Figure 3 for illustration. ■

The rest of the proof uses essentially the same arguments as for the proof of 2-degenerate graphs. The only difference is that we can always find a vertex having at most 1 high degree neighbor. Thus, for each newly added edge, the maximum number of unavailable colors is $2 \times \Delta + (d(v) - 2) \times 2 \leq 4\Delta - 4$. Thus setting $\mathcal{B} = \{1, 2, \ldots, 4\Delta - 3\}$ and $\mathcal{A} = \{1', 2', \ldots, (4\Delta - 3)'\}$, respectively, as the base and additional sets of colors, the induction steps goes through and the result follows. ■

3. ALGORITHMIC ASPECTS, CONCLUSION, AND REMARKS

Though we do not explicitly give an algorithm, our proof arguments are constructive and inherently algorithmic. The algorithm is implicit in the arguments. Given a 2-degenerate graph G, an ordering of the edges such that the graph can be constructed from the trivial graph can be obtained efficiently. Since we do not try to analyze the complexity, a safe upper bound seems to be $O(|V|\Delta|E|^2)$. Once the ordering is available, none of the $|E|$ coloring steps take more than $O(\Delta)$ probes.

We have shown that for a 2-degenerate graph, the strong chromatic index is linear in terms of the maximum degree. We make use of a structural result on k-degenerate graphs for the case $k = 2$. We believe that this result can be extended to arbitrary k. That is, we conjecture the following.

Conjecture 8. *There exists an absolute constant c such that for any k-degenerate graph G,* $\chi'_s(G) \le ck^2\Delta$. *Thus for fixed k,* $\chi'_s(G)$ *is linear in* Δ.

In fact, we feel that something even stronger may be true and the exponent of k can be made 1. Writing a bit speculatively, the main difficulty that arises for arbitrary k seems to be in dealing with the situation when already primed edges are made adjacent. It may be possible to address this by using a list of k^2 alternatives associated with each color.

Note that, in our proofs, we double the colors in one step to avoid some technical difficulty that may be possibly avoided, though we do not see a direct way yet. We leave it as an open problem.

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REFERENCES

[1] B. Bollobás, Extremal Graph Theory, Dover, 1978, Academic Press: New York.

[2] V. Borozan, L. Montero and N. Narayanan. Further results on strong edge colouring. Manuscript 2013

[3] R. A. Brualdi and J. J. Quinn Massey, Incidence and strong edge colorings of graphs, Discrete Math 122(1–3) (1993), 51–58.

[4] G. A. Dirac, Minimally 2-connected graphs, J Reine Angew Math 228 (1967), 204–216.

[5] R. J. Faudree, A. Gyárfás, R. H. Schelp, and Z. Tuza, Induced matchings in bipartite graphs, Discrete Math 78(1–2) (1989), 83–87.

[6] R. J. Faudree, A. Gyárfás, R. H. Schelp, and Z. Tuza, The strong chromatic index of graphs, Ars Combin 29(B) (1990), 205–211.

[7] B. Lèvêque, F. Maffray, and N. Trotignon, On graphs with no induced subdivision of K_4, http://www.liafa.jussieu.fr/∼trot/articles/isk4.pdf.

[8] R. Luo and G. Yu. A note on strong edge colouring of 2-degenerate graphs. http://arxiv.org/abs/1212.6092

[9] M. Mahdian, The strong chromatic index of C_4-free graphs, Random Stru Algor 17 (2000), 357–375.

[10] M. Molloy and B. Reed, A bound on the strong chromatic index of a graph, J Combin Theory Ser B 69 (1997), 103–109.
[11] R. Muthu, N. Narayanan, and C. R. Subramanian, Some graph classess satisfying acyclic edge colouring conjecture, http://www.imsc.res.in/∼narayan/acyc-hnk2.pdf.
[12] K. Nakprasit, A note on the strong chromatic index of bipartite graphs, Discrete Math 308 (2008), 3726–3728.
[13] M. D. Plummer, On minimal blocks, Trans Amer Math Soc 134(1) (1968), 85–94.
[14] J. J. Quinn and A. T. Benjamin, Strong chromatic index of subset graphs, J Graph Theory 24 (1997), 267–273.
[15] T. K. Šumenjak, Θ-graphs of partial cubes and strong edge colorings, Elect Notes Discrete Math 28 (2007), 521–526.