Evaluation of sphericity error from form data using computational geometric techniques

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Abstract

The measurement data for evaluation of sphericity error can be obtained from inspection devices such as form measuring instruments/set-ups. Due to misalignment and size-suppression inherent in these measurements, sphericity data obtained will be distorted. Hence, the sphericity error is evaluated with reference to an assessment feature, referred to as a limacoid. Appropriate methods based on the computational geometry have been developed to establish Minimum Circumscribed, Maximum Inscribed and Minimum Zone Limacoids. The present methods start with the construction of 3-D hulls. A 3-D convex outer hull is established using computational geometric concepts presently available. A heuristic method is followed in this paper to establish a 3-D inner hull. Based on a new concept of 3-D equi-angular line, 3-D farthest or nearest equi-angular diagrams are constructed for establishing the assessment limacoids. Algorithms proposed in the present work are implemented and validated with the simulated data and the data available in the literature. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Sphericity error; Computational geometry; Equi-angular diagram; Minimum circumscribed limacoid; Maximum inscribed limacoid; Minimum zone limacoids

1. Introduction

Spherical components such as contact probes in inspection devices, balls in bearings and other precision products need to be manufactured and inspected with strict dimensional and geometric controls to achieve high levels of functional performance. The geometrical imperfections such as surface roughness, waviness and form error have remarkable influence on performance of these components. For example, the imperfections on the precision engineering components of gyroscopes and measuring instruments will lead to erroneous information [9,2]. In heavily loaded components like ball bearings in engines, turbines etc., these imperfections may result in the generation of a large amount of heat and in turn lead to reduction in life. Hence it is important to carefully evaluate the spherical surfaces for geometric form imperfections.

The International Standard ISO 3290 [5] dealing with rolling bearings, bearing parts and balls for rolling bearings, defines the deviation of spherical form as the greatest radial distance in any radial plane between a sphere circumscribing the ball surface and any point on the ball surface. As practised in rolling bearing industries, the ball profiles are obtained in two or three equatorial planes at 90° to each other using a roundness measuring instrument. The evaluation based on roundness measurements, namely two-dimensional measurements, is inadequate to arrive at the sphericity error that is three-dimensional in nature. The sphericity data must be obtained using appropriate inspection devices such as form measuring instruments/set-ups [6]. The data must also be processed using appropriate techniques to obtain reliable results during the evaluation of sphericity error. It is also important that these techniques have to follow the specifications laid down in the standards. International Standard ISO 1101 [4] does not deal with the sphericity tolerance explicitly and therefore the concepts of form tolerances dealt with in this standard have to be extended to the evaluation of the sphericity error.

Many attempts have been made to evaluate circularity
error from the form data obtained using roundness measuring instruments. Since the roundness profile is distorted by the misalignment between the axes of a component and a rotating table and by radius-suppression inherent in the measurement, a limacon is used as an assessment feature [15]. Using a limacon as the assessment feature, the Least Squares Method (LSM) and several numerical methods have been employed to arrive at the circularity error. Apart from Minimum Zone evaluation, the concepts of Minimum Circumscribed and Maximum Inscribed limacons are also used in circularity evaluation. Considering the limacon as a curve representing the locus of foot of the perpendiculars drawn from a pole to the tangents of a circle, the computational geometric techniques have been successfully developed to arrive at the circularity error [17].

In the case of sphericity, a few attempts have been made by previous researchers to develop methods for evaluating the sphericity error from the form data. The Least Squares Method that minimizes the sum of the squared deviations of the measured points from a fitted feature has been suggested [15]. Though the Least Squares Method is based on sound mathematical principles, the error values obtained are not the minimum [10,11]. A statistical technique for estimating the sphericity from a few 2-D roundness values was also proposed [7]. Shunmugam has suggested a new simple approach called the Median technique that gives minimum value of sphericity error [15]. Using discrete Chebyshev approximations, Danish and Shunmugam have arrived at the minimum zone values [1]. Apart from the Minimum Zone principle, the concepts of Minimum Circumscribed and Maximum Inscribed features used in circularity evaluation can also be extended to the evaluation of the sphericity error.

In the past decade, the computational geometric techniques have gained enormous attention from the designers of algorithms for solving geometric problems. These computational geometry based algorithms can be very well applied for the evaluation of the sphericity error in manufactured components. Computational geometry based techniques show greater promise for solving the minimum zone problems encountered in the geometrical evaluations. However, no attempt has been made by researchers in the past to adopt computational geometry based assessment techniques for different methods of evaluation of the sphericity error. In this paper, techniques for evaluation of the sphericity error using form data have been presented. To take care of the distortion introduced during the sphericity measurement, a 3-D assessment feature referred to as a limacoid is established by extending the concept of a limacon. Apart from using the existing computational geometric concept of a 3-D convex outer hull, a heuristic method for obtaining a 3-D convex inner hull has been proposed. A new concept of 3-D equi-angular line is introduced and 3-D farthest and nearest equi-angular diagrams (EA diagrams) are developed to arrive at different assessment limacoids. The results obtained using the simulated data and data reported in the literature are included in this paper.

### 2. Evaluation of sphericity error

Fig. 1 shows a schematic arrangement for the measurement of spherical form. As only the deviation from the true form is measured, there will be size suppression (radius suppression) in the form data. The data obtained from the measuring instruments/set-ups will be in the polar form, with the radial values in microns and the angles in degrees. Also, the misalignment between component center and the axis of the measuring instrument/setup cannot be totally eliminated. Due to the misalignment and size suppression, the form data obtained will be distorted. Even when a truly spherical surface is measured, if the axis of the rotating table of the measuring instrument does not coincide with the component center, distortion will be introduced in the measurement data. The surface obtained by such measurement is approximated to a new surface referred

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**Nomenclature**

- \( r_i, \theta_i, \beta_i \): Polar coordinate of a point
- \( x_0, y_0, z_0 \): Estimated coordinates of center of the assessment feature
- \( r_0 \): Radius of assessment sphere
- \( r_a \): Radius of the assumed sphere
- \( e_i \): Deviation of \( i \)th point from assessment feature
- \( e_{\text{max}}, e_{\text{min}} \): Maximum and minimum deviations of the profile
- \( \Delta \): Sphericity error
- \( r_i^{'}, r_{\text{max}}, r_{\text{min}} \): Radius of the points on inverted profile
- \( \delta \): Increment
- \( k \): Constant, from 0.5 to 1.0
Fig. 1. Schematic arrangement for spherical form measurement.

to as a limacoid in this paper. The limacoid is a surface that passes through the foot of the perpendicular drawn from a pole to the tangent planes of a given sphere. The geometric properties and the construction of the limacoid are given in Appendix A. The limacoid is also expressed in the linear form as

$$r_i = r_0 + x_0 \cos \beta_i + y_0 \cos \theta_i + z_0 \sin \beta_i$$

(1)

For form data shown in Fig. 2, the sphericity error is evaluated with reference to the assessment limacoid and the deviation of a point $$(r_i, \theta_i, \beta_i)$$ and are expressed in the linear form as

$$e_i = r_i - (r_0 + x_0 \cos \beta_i + y_0 \cos \theta_i + z_0 \sin \beta_i)$$

(2)

where $$x_0, y_0, z_0$$ refer to the center and $$r_0$$ is the radius of the sphere from which the limacoid is obtained [16]. The sphericity error with reference to the assessment feature is given by

$$\Delta = |e_{\text{max}}| + |e_{\text{min}}|$$

(3)

where $$e_{\text{max}}$$ and $$e_{\text{min}}$$ are the maximum and minimum deviations of the data points from the assessment feature. $$e_{\text{max}}$$ is zero for a Minimum Circumscribed Limacoid (MCL) and all the points will be enclosed by this feature. $$e_{\text{min}}$$ is zero in the case of a Maximum Inscribed Limacoid (MIL) and the points lie outside. For Minimum Zone (MZ) evaluation, the deviation can be computed as the least value obtained when a pair of limacoids has same center for the spheres from which they are developed and contain all the points in the annular space between them.

3. Algorithms for evaluation of sphericity

3.1. Minimum Circumscribed Limacoid (Crest Limacoid)

The Minimum Circumscribed Limacoid (MCL) will pass through four vertices of the outer hull. Keeping this in view, a 3-D convex outer hull is constructed first for the given set of data points to eliminate most of the points that lie inside and do not influence the final result. The 3-D convex outer hull (sometimes referred to as the outer hull) of a given set of points in Euclidean space

Fig. 2. Spherical form data ($$r_i$$ in \(\mu\text{m}, \theta_i$$, and $$\beta_i$$ in degrees) for illustration only.
is the boundary of the smallest convex domain containing all the points of the set. A domain is said to be convex if, for any two points in the domain, the segment connecting them should be entirely contained in the domain [14]. In other words, the convex hull of points in three-dimensional space is the shape taken by plastic wrap stretched tightly around the points [12]. The algorithms used in the present work for constructing the convex hull is based on the Divide-and-Conquer and Merge techniques suggested by Preparata and Hong [13].

Fig. 3(a) shows the outer convex hull with vertices $V_1$, $V_2$ etc. for the data points shown in Fig. 2. In the next step, the initial EA lines are established. The details of the EA planes and lines are given in Appendix B. Taking the vertices $V_2$, $V_3$ and $V_4$, lines connecting these vertices to the origin $O$ are obtained. Three planes $PL_2$, $PL_3$ and $PL_4$ perpendicular to these lines $V_2O$, $V_3O$ and $V_4O$ are established as shown by thin lines in Fig. 3(a). For each pair of planes, a plane bisecting the angle between them (referred to as the EA plane) is established. There will be three such EA planes and the intersection of these three EA planes results in an EA line. For the three planes $PL_2$, $PL_3$ and $PL_4$, the EA line obtained is referred to as $EA_{234}$. Taking any point on this EA line, a sphere can be drawn tangential to the three planes $PL_2$, $PL_3$ and $PL_4$. In the next step, the farthest center of the EA line is determined. For example, $EA_{234}$ will be intersected by three adjacent EA planes (corresponding to $PL_2$ and its adjacent plane, $PL_3$ and its adjacent plane, etc.). The farthest intersection point among them gives the farthest center and the sphere drawn with it as a center will also be tangential to the adjacent EA plane resulting in this intersection point. The portion of the EA line beyond the farthest center is known as the farthest edge, $FE_{234}$. Following the above procedure, all EA lines and farthest edges are constructed. The initial farthest EA edges are shown in Fig. 3(b).

After constructing the initial EA diagram, the hull is updated by eliminating the vertices with adjacent faces having common farthest EA centers. Different cases that may be encountered and updating of the hull in these cases are shown in Fig. 4. Fig. 4(a) shows three points of a face $V_1$, $V_2$, and $V_3$. Radial lines are established through these points and the planes $PL_1$, $PL_2$ and $PL_3$ are established tangential to the radial lines through $V_1$, $V_2$ and $V_3$, respectively. The EA line for these planes is established following the procedure explained in Appendix B. Fig. 4(b (i)) shows three faces formed by four vertices $V_1$, $V_2$, $V_3$ and $V_4$, and all have a common farthest center. Any two faces can also have a common farthest center as shown in Fig. 4(b (ii)). In both cases, the vertex $V_1$ is deleted and the hull is updated by forming a new face $V_2$, $V_3$, $V_4$, as shown in Fig. 4(b (iii)).

In the case of a vertex with four faces, if all the faces have a common farthest center as shown in Fig. 4(c (i)), the common vertex $V_1$ is deleted. The updated faces are obtained either by joining $V_2$ and $V_4$ or $V_3$ and $V_5$. If there is at least a pair of adjacent faces with a common farthest center, the edge common to adjacent faces is eliminated. For example, Fig. 4(c (ii)) shows that two faces $V_1$, $V_3$, $V_4$, and $V_1$, $V_4$, $V_5$ have a common center $C_{1345}$ and faces $V_1$, $V_2$, $V_3$ and $V_1$, $V_2$, $V_5$ have another common center $C_{1235}$. In this case also, the vertex $V_1$ along with the edges $V_1$, $V_4$ and $V_1$, $V_2$ gets eliminated resulting in the formation of a new edge $V_3$, $V_5$ as shown in Fig. 4(c (iii)). For other cases with more than four faces also, the same logic is applied and the updated faces are obtained. The updating procedure is equivalent to the formation of faces in arriving at the convex hull.

The first stage of updating is carried out by considering the vertices and faces that have common centers. However, a vertex belonging to the updated face should not be deleted in the process of updating the vertices and the faces in the neighborhood. After the first stage of updating covering all potential vertices, the next stage of updating starts. The process of updating continues till the number of vertices is reduced to three.

The hull shown in Fig. 3(b) with the initial farthest EA edges is updated following the procedure explained above. The vertices $V_1$ and $V_3$ get eliminated as the three faces surrounding each of these vertices have a common center. The remaining vertices are $V_2$, $V_4$ and $V_5$. The EA line for this face results in $FE_{245}$, as shown in Fig. 5. The centers obtained at all stages of updating are the candidate centers for the spheres from which the circumscribed limacoid can be generated. A complete EA diagram with Minimum Circumscribed Limacoid is shown in Fig. 5. This Minimum Circumscribed Limacoid passes through four crest points and encloses all other points of the data set, in such a way that the sphere from which it is generated has the least radius.

### 3.2. Maximum Inscribed Limacoid (Valley Limacoid)

The points controlling the inscribed limacoid lie on the inner hull and, therefore, the inner hull of the given set of data points is determined before constructing the nearest EA diagram. The inner hull of a set of points is the boundary of the largest empty subset of a given set of points.

A new heuristic algorithm for finding the inner hull is proposed in the present work. This method involves transformation of the original data with reference to the measurement center $O$ of the sphere as shown in Fig. 6(a). With this center, a suitable diameter of the sphere is selected such that all the measured points are within this assumed sphere. The transformation is done by arriving at radial deviations between the measured points and the corresponding points on the assumed sphere. For example, the point $P_1$ is obtained by taking along the
corresponding radial line the distance $a$ from the center $O$, that represents the deviation $P_1S_1$.

Assuming a very large radius $r_a$ can lead to considerable distortion of the data. A simple heuristic rule to select the radius of this sphere is

$$r_a = r_{\text{max}} + \delta$$  \hspace{1cm} (4)

where $\delta = k(r_{\text{max}} - r_{\text{min}})$ and $k$ can be selected between 0.5 and 1.0. $r_{\text{max}}$ and $r_{\text{min}}$ are the maximum and minimum radii with respect to the selected center.
The 3-D convex outer hull for the transformed data is determined as explained earlier. The points of the original data set corresponding to the vertices of the outer hull of the transformed data are taken as the vertices of the inner hull as shown in Fig. 6(b) schematically. The procedure for constructing the nearest EA diagram is similar to that of the farthest EA diagram, except that the nearest common centers are considered as end points of the nearest EA lines. Fig. 6(b) shows the initial nearest EA lines for the faces of the inner hull. For example, the nearest EA line NE_{123} for face V_1, V_2, V_3 is obtained by finding the EA center nearest to the face. The hull is updated in a similar way as explained earlier. The vertices V_3 and V_4 are eliminated as they have common nearest EA centers. The nearest EA line NE_{125} is established for the remaining three vertices of the hull V_1, V_2 and V_5. The complete nearest EA diagram and the Maximum Inscribed Limacoid are shown in Fig. 6(c). The Maximum Inscribed Limacoid passes through four points and all other points lie outside. The sphere from which it is generated has a maximum radius.
3.3. Minimum zone limacoids

The minimum zone value is obtained when a pair of limacoids generated from two concentric spheres contain all the points of the data set and have minimum distance between them.

In order to determine the minimum zone limacoids, the farthest and nearest EA diagrams are superimposed as shown in Fig. 7. The following cases may arise, while considering candidate centers for minimum zone error. (i) All the centers of the farthest and nearest EA diagram may be the candidate centers. When these centers are selected, the inner or outer assessment limacoid will pass through four points and the other limacoid through at least one point of the data set. (ii) The intersection points of farthest and nearest EA edges may be taken as the candidate centers. In this case, both circumscribed and inscribed limacoids will pass through three points each. However, when an intersection point coincides with the centers of the farthest or nearest EA diagram, four points lie on one of the limacoids and the other limacoid will pass through three points. (iii) Candidate centers may also be obtained as the intersection of EA plane for two points (Refer to Fig. 9 in Appendix B) and EA line for three points (Refer to Fig. 10 in Appendix B). In this case, three points lie on the circumscribed or inscribed limacoid and two points on the other limacoid. If such intersection points are considered for all the points of the data set, the number of intersection points will be very large and it will take a longer time for computation. As the potential minimum zone control points will be among the crest and valley points, the intersection points of the EA line and the EA plane for these points alone are considered as the candidate centers for minimum zone evaluation.

The radial distance between the spheres from which the limacoids are generated is determined for all the candidate centers and a pair that has minimum separation is taken to represent the Minimum Zone Limacoids. The sphericity error is obtained from these Minimum Zone (MZ) Limacoids.

4. Results and discussion

The computer programs were written in C++ based on the proposed algorithms. The package was run on a Pentium III, 128 MB, 450 MHz machine. The programs were tested for the simulated data and the data available in the literature. The data sets are shown in Table 1 and Table 2. The algorithms developed in the present work are validated with the published results. Table 3 shows the results obtained on the basis of Minimum Circumscribed Limacoid (MCL), Maximum Inscribed Limacoid (MIL) and Minimum Zone (MZ) Limacoids. For the purpose of comparison, the values based on the Least Squares (LS) method are also included in Table 3. The LS solution for the form data is found by minimizing the sum of the squares of the deviations given by Eq. (2). The LS technique yields larger values of sphericity error.

The evaluation of the sphericity error by the computational geometry technique yields superior results with error values equal to or less than the results published in the literature as shown in Table 3. For the form data given in Data Set 1, the reported value of sphericity error is 2.88 µm [15], whereas the present method gives minimum zone error of 2.82843 µm. This is in agreement with the error value reported by Danish and Shunmugam [1]. However, the numerical methods reported in the literature for the evaluation of sphericity error based on crest and valley features require some value for convergence. Though the MZ solution can be obtained directly by certain numerical methods such as the Chebyshev approximation [1], some attempts have been made to use other methods that require a convergence value [8]. In all these cases, the accuracy of the solutions obtained depends on the value specified for the convergence. The algorithms proposed in the present work does not require any specified convergence value and the results depend only on the points that are used for establishing the assessment limacoids.

The form data set given in Table 2 is simulated from CMM data given by Fan and Lee [3]. For this purpose, a suitable transformation is applied as reported by the authors elsewhere. It is found that the minimum zone error value obtained for this data set by the present method is well in agreement with the reported value of 7.66 µm and the data points 7, 16 and 45 lie on the outer limacoid and the points 20 and 39 on the inner one.
In the proposed algorithms, the convex outer hull and the inner hull for the given set of data points are considered so that the points which do not influence the final results are eliminated and the construction of EA diagrams becomes less cumbersome. However, care should be taken while finding the inner hull and the subsequent construction of the EA diagram.

The computational complexity of finding the convex outer hull by a divide and conquer algorithm is $O(n \log n)$. The complexity of finding the inner hull is also $O(n \log n)$ as the time required for transforming the data is $O(n)$, and for finding the outer hull of the transformed data is $O(n \log n)$. The EA diagrams can be constructed in $O(n)$ time. Hence the overall complexity of the proposed algorithms is $O(n \log n)$.

5. Conclusions

Considering the need for fast and efficient algorithms for processing the geometric data obtained from the form measuring instruments, the algorithms based on computational geometric techniques have been developed. It is established that the limacoids are assessment features to be considered for evaluating the sphericity error using form data. The geometry of a limacoid has been explained for the first time by the authors and this has helped in the development of the appropriate computational geometric techniques. New concepts involving EA diagrams have been introduced and successfully applied in this work.

The minimum zones as well as Minimum Circumscribed and Maximum Inscribed methods of evaluation
of the sphericity error using form data have been carried out and the results are presented. In the present work, the effectiveness of the computational geometry based approaches in the evaluation of sphericity error has been demonstrated successfully using simulated data and the data available in literature.

A heuristic method proposed in this work for obtaining the 3D-inner hull works well for the evaluation of the sphericity error by the MIL method. The transformation of the data suggested in this heuristic method has to be carried out carefully to maintain the nature of

\[ r_i, \mu_i, \theta_i, \beta_i \]

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Table 3
Sphericity error using form data

<table>
<thead>
<tr>
<th>Data Parameters</th>
<th>Reported method</th>
<th>Present method</th>
<th>LS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Crest Valley</td>
<td>MZ</td>
<td>MIL</td>
</tr>
<tr>
<td>Set 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_0 (\mu m)$</td>
<td>1.29 c</td>
<td>1.59 c</td>
<td>1.41 b</td>
</tr>
<tr>
<td>$y_0 (\mu m)$</td>
<td>0.296 c</td>
<td>0.17 c</td>
<td>0.24 b</td>
</tr>
<tr>
<td>$z_0 (\mu m)$</td>
<td>0.71 c</td>
<td>0.728 b</td>
<td>0.58 b</td>
</tr>
<tr>
<td>$r_0 (\mu m)$</td>
<td>4.29 c</td>
<td>1.62 c</td>
<td>3.00 b</td>
</tr>
<tr>
<td>$\Delta (\mu m)$</td>
<td>2.88 b</td>
<td>2.97 c</td>
<td>2.82 b</td>
</tr>
<tr>
<td>Control points</td>
<td>1.14,20,23 c</td>
<td>3.6,19,24 b</td>
<td>1.6,14,19,24, 1.14,20,23 c</td>
</tr>
</tbody>
</table>

Set 2 $x_0 (\mu m)$       –       –       –       1.86667     0.63818      1.11808     –0.53055
$y_0 (\mu m)$       –       –       –       0.50951     0.22282      0.41494     0.13803
$z_0 (\mu m)$       –       –       –       –0.35705    –0.60176     –0.17266     0.12931
$r_0 (\mu m)$       –       –       –       9.25327     1.78505      5.37178     4.76377
$\Delta (\mu m)$       –       –       –       8.391589    7.85256      7.66012     8.48576
Control points       –       –       –       2.7,16,45    20.25,39,48  7.16,20,39,45 –

a Assessment feature having equal magnitude of maximum deviations on either side.
b Danish and Shunmugam ([1]).
c Shunmugam ([15]).

the surface measured. The present method for minimum zone evaluation always guarantees a minimum value for a given set of data points.

These algorithms are computationally quite robust giving unique solutions and require a short time for execution. The proposed algorithms can handle data with both uniform and non-uniform spacing. The present algorithms cannot only be applied to measurement data corresponding to a full sphere but also to the partial sphere.

Appendix A. Limacoid and its properties

A limacoid has a pole O and an axis OO o. In Fig. 8(i), a base sphere is drawn with OO o as its diameter. A point P i on the limacoid is at a distance of $r_i$ from the pole where the distance between the point P i and P j located on the base of a sphere is equal to a constant value $r_o$. Similarly, other points on the limacoid are obtained by laying out the constant value $r_o$ from the corresponding points on the sphere. The surface of the limacoid passes through all these points.

Alternatively, a sphere of radius $r_o$ is drawn with O o as its center, as shown in Fig. 8(ii). A plane is established passing through a point T on the sphere and tangent to it. From the pole O, a perpendicular line to this tangent plane is established. The foot of this perpendicular gives P i that lies on the limacoid. The distance of this point from the pole is given by $r_i$. The limacoid is obtained as a surface that passes through the foot of the perpendicular to the tangent planes of the sphere.

Fig. 8(iii) shows two points P 1 and P 2 on the limacoid and the corresponding points T 1 and T 2 on the sphere. It can be inferred that the line OP i and O o T i are perpendicular to the tangent plane established at point T i of the sphere. O o T i is also a radial line for the sphere. For the second point T 2 on the same sphere, there will be another set of lines OP 2 and O o T 2, perpendicular to the second tangent plane. By geometry, it can be shown that the plane bisecting the angle between the first and second tangent planes will pass through the center of the sphere. Alternatively, if a plane is established at P i perpendicular to the line joining P i and the pole O and another plane is established in a similar way at a different point P 2 on the limacoid, then the angle bisector of these two planes will also pass through the center of the sphere from which the limacoid is established.

Appendix B. Equi-angular planes and lines

Fig. 9 shows two planes PL 1 and PL 2 intersecting. The equi-angular plane (EA plane) is drawn such that it bisects the angle between the two planes PL 1 and PL 2. For any point X on the EA plane, the perpendicular distance d to the planes PL 1 and PL 2 will be equal.

Three planes PL 1, PL 2 and PL 3 intersecting each other are shown in Fig. 10. The equi-angular plane EA 12 bisects the angle between the planes PL 1 and PL 2. Similarly the EA planes EA 23 and EA 13 are established. These three EA planes EA 12, EA 23 and EA 13 intersect along a line. This line will be referred to as the EA line hereafter. Any point X on this EA line will be at an equal distance from the three planes and this line will pass through the point of intersection of the three planes PL 1, PL 2 and PL 3. A sphere can be drawn with X as a center such that it is tangential to the planes PL 1, PL 2 and PL 3.
Fig. 8. Geometry of a limacoid.

Fig. 9. Equi-angular plane.

Fig. 10. Line intersection of three EA planes.

Fig. 11 shows the three planes PL₁, PL₂ and PL₃ along with the adjacent planes PL₄, PL₅ and PL₆. The line EA₁₂₃ corresponds to the three planes PL₁, PL₂ and PL₃ as explained earlier. This line EA₁₂₃ is intersected by three other EA planes, namely EA₁₄, EA₂₅ and EA₃₆, established taking the adjacent planes PL₄, PL₅ and PL₆. Three intersection points are obtained. The intersection point Cₕ farthest from the planes PL₁, PL₂ and PL₃ is the farthest center and intersection point Cₙ nearest to the planes is the nearest center. For example, if the farthest center Cₕ is obtained by intersection of the EA line EA₁₂₃ with the plane EA₁₄, a largest possible sphere can be drawn tangential to the four planes PL₁, PL₂, PL₃ and PL₄ with Cₕ as a center. The line beyond Cₕ is known as farthest EA line FE₁₂₃. Any point taken along the line EA₁₂₃ away from Cₕ can yield a sphere tangential to the three planes PL₁, PL₂ and PL₃. The portion of the line from Cₙ towards the planes PL₁, PL₂, PL₃ is known as nearest EA line NE₁₂₃ and point Cₙ would result in the smallest sphere.
Fig. 11. Farthest and nearest centers/edges.

References


