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Synopsis

Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field $k$ of characteristic 0 and let $P$ be a projective $A$-module of rank $n$. It is well-known that in general $P$ may not split off a free summand of rank 1 as $C_n(P) = 0$ is a necessary condition, where $C_n(P)$ denotes the $n^{th}$ Chern class of $P$ which is an element of the group $\text{CH}_0(X)$ of zero cycles modulo rational equivalence. A result of Murthy [18, Theorem 3.8] says that when $k$ is an algebraically closed field, $C_n(P) = 0$ is also sufficient.

However, if $k$ is not algebraically closed, then $C_n(P) = 0$ is not always a sufficient condition as evidenced by the example of the module of differentials $\Omega_{A/\mathbb{R}}$ where $A$ denotes the coordinate ring of the even dimensional real sphere. Hence, it is of interest to know when $C_n(P) = 0$ is a sufficient condition for $P$ to split off a free summand of rank 1.

To tackle this problem, we look towards topology for intuition, where there is a very well-developed obstruction theory to deal with the problem of when a vector bundle has a nowhere vanishing section. When $\mathcal{E}$ is a smooth, oriented, real vector bundle of rank $n$ over a smooth, oriented $n$-dimensional manifold $X$, we can associate to $\mathcal{E}$ its Euler class $e(\mathcal{E}) \in H^n(X, \mathbb{Z})$. Then it is known that $\mathcal{E}$ has a non-vanishing section (or equivalently splits off a free summand of rank 1) if and only if $e(\mathcal{E}) = 0$.

Through the early nineties, there has been a body of work that defines Euler classes for a projective module $P$ over a commutative Noetherian ring where rank $n = \text{dim}(A)$. These classes take values in the Euler class group of $A$, which was introduced by Nori ([6]). The Euler class turns out to be a finer invariant than the top Chern class when the ring is a smooth, affine algebra of dimension
\[ n \geq 2 \] over \( k \) where characteristic of \( k \) is 0. Using this theory, Bhatwadekar and Raja Sridharan could answer the question about whether the vanishing of the top Chern class is a sufficient condition in the case when \( k = \mathbb{R} \) for “oriented” projective modules of top rank in [5] and provided a complete characterisation of when this happens. Subsequently, Bhatwadekar, Das and Mandal could remove the condition of orientedness and get a general classification in [2].

In this thesis, we extend these results to the case when the base field \( k \) is a real closed field ([8]). We state our main result:

**Theorem B.** Let \( \mathbb{R} \) be a real closed field. Let \( X = \text{Spec}(A) \) be a smooth affine variety of dimension \( n \geq 2 \) over \( \mathbb{R} \). Let \( X(\mathbb{R}) \) denote the \( \mathbb{R} \)-rational points of the variety. Let \( K \) denote the module \( \Lambda^n(\Omega_A/\mathbb{R}) \). Let \( P \) be a projective \( A \)-module of rank \( n \) and let \( \Lambda^n(P) = L \). Assume that \( C_n(P) = 0 \) in \( CH_0(X) \). Then \( P \cong A \oplus Q \) in the following cases:

1. \( X(\mathbb{R}) \) has no closed and bounded semialgebraically connected component.

2. For every closed and bounded semialgebraically connected component \( W \) of \( X(\mathbb{R}) \), \( L_W \not\cong K_W \) where \( K_W \) and \( L_W \) denote restriction of (induced) line bundles on \( X(\mathbb{R}) \) to \( W \).

3. \( n \) is odd.

Moreover, if \( n \) is even and \( L \) is a rank 1 projective \( A \)-module such that there exists a closed and bounded semialgebraically connected component \( W \) of \( X(\mathbb{R}) \) with the property that \( L_W \cong K_W \), then there exists a projective \( A \)-module \( P \) of rank \( n \) such that \( P \oplus A \cong L \oplus A^{n-1} \oplus A \) (hence \( C_n(P) = 0 \)) but \( P \) does not have a free summand of rank 1.

Using the already existing result in [2], Tarski’s principle and the Artin-Lang homomorphism theorem, the result in [2] can be extended to Archimedean real closed fields as we have done in [7]. **Theorem A** extends the result to arbitrary real closed fields and we use neither the earlier result in [2] nor Tarski’s principle in the proof. Further, the proof in [2] uses topological arguments along with the key topological fact that intervals in \( \mathbb{R} \) are connected. Since such methods are unavailable in the case where \( k = \mathbb{R} \) is an arbitrary real closed field (in fact
“intervals” there are totally disconnected), our arguments are completely algebraic (modulo whatever topology appears in the statement itself). Thus, Theorem A is an algebraic generalisation of the result in [2].

The result (and the algebraic method of proof) gives further evidence to hope for an affirmative answer to the following question of Bhatwadekar, Das and Mandal:

**Question.** Let $X = \text{Spec}(A)$ be a smooth affine variety of odd dimension $n \geq 2$ over a field $k$ of characteristic 0. Let $P$ be a projective $A$-module of rank $n$ such that $C_n(P) = 0$ in $\text{CH}_0(X)$. Then, does there exist a projective $A$-module $Q$ of rank $n-1$ such that $P \simeq Q \oplus A$?

Let $R, X, A, L$ be as in Theorem B. Let $R(A)$ denote the ring of real regular functions, i.e. the ring obtained by inverting all elements which do not belong to any real maximal ideal and let $R(L) = L \otimes_A R(A)$.

As in [2] and [7], the key ingredient in the proof of Theorem B is a structure theorem for the Euler class group $E(R(A), R(L))$. This structure theorem is in terms of the semialgebraically connected semialgebraic components of the space of real points $X(R)$.

**Theorem A.** Let $A, K, L, R(A)$ be as above. Let $C_i, 1 \leq i \leq t$ be the closed and bounded semialgebraically connected semialgebraic components of $X(R)$. Let $L_i$ and $K_i$ be the restriction of the semialgebraic line bundles corresponding to $L$ and $K$ respectively, to $C_i$. Let $L_i \simeq K_i$, for $1 \leq i \leq r$ and $L_i \not\simeq K_i$, for $r+1 \leq i \leq t$. Let $x_i \in C_i$ and let $\mathcal{M}_i$ be the corresponding maximal ideal of $R(A)$. Let $\omega_i$ be a local $R(L)$-orientation of $\mathcal{M}_i$. Then, $\bigoplus_{i=1}^{r} \mathbb{Z} e_i \bigoplus_{i=r+1}^{t} (\mathbb{Z}/2)e_i \sim E(R(A), R(L))$ sending $e_i \mapsto (\mathcal{M}_i, \omega_i)$ is an isomorphism.
Chapter 1

Introduction

1.1 Some History and Intuition

Let $A$ be a commutative Noetherian ring of dimension $n$. We are interested in studying finitely generated projective modules over this ring. In particular, we want to consider the question of when a projective module splits off a free summand of rank 1. This question originates, amongst other places from the problem of finding nowhere vanishing sections of vector bundles in topology. Indeed, the module of sections of a vector bundle is a projective module over the ring of continuous functions (refer [23] for a more detailed exposition of this connection). It is a well-known result in topology that a vector bundle with rank larger than the dimension of the base space has a nowhere vanishing section, which also means it splits off a trivial line bundle. This motivates the following result of Serre.

**Theorem 1.1.1** (Serre). For any commutative, Noetherian ring $A$ and an $A$-projective module $P$, if $t = \text{rank}(P) > \dim(A) = n$, then

$$P \cong A^{t-n} \oplus Q.$$  

Of course if the ring $A$ is a principal ideal domain (p.i.d.), then all projective modules are free, thanks to the well-known structure theorem for finitely generated modules. The first examples of non-free projective modules occur
when $A$ is a Dedekind domain which is not a p.i.d. In this case, there is an ideal which is not principally generated, and this ideal furnishes the example of a finitely generated projective module which is not free. Historically, these examples are given by the so-called “ring of integers” in number theory or smooth curves in geometry, e.g. the ideal $(2, 1 + \sqrt{-5})$ in the ring $\mathbb{Z}[\sqrt{-5}]$ or the ideal $(X - 1, Y)$ of $\mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$. These examples show that without any further assumptions, Serre’s result is the best possible.

It is natural then to look at vector bundles and what subsequent theory we have for them, particularly for results in the case where the rank equals the dimension of the base space of the bundle.

### 1.2 Obstruction Theory

Serre’s result (1.1.1) implies that the original question about a projective module splitting off a free summand always has an affirmative answer when the rank of the projective module is larger than the dimension of the ring. Hence, the first non-trivial case under consideration is that of projective modules which have the same rank as the dimension of the ring.

In line with obstruction theory for vector bundles, there are naturally defined classes called Chern classes which take values in the Chow groups of the corresponding ring $A$, i.e. $c_i(P) \in CH^i(A)$. Then, it is easy to prove from their properties that :

**Theorem 1.2.1.** $P \simeq A^r \oplus Q \Rightarrow c_j(P) = 0 \forall j > n - r$.

Thus, these classes are obstructions to splitting. Note that $CH^n(A)$ is same as $CH_0(A)$, the group of zero cycles modulo rational equivalence (refer section 2.1 for details). The above result specialised to $j = n$ in particular implies that $c_n(P) \neq 0$ in $CH_0(A)$ implies $P$ does not split off a free summand of rank 1.

Now, let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field $k$ of characteristic 0 and let $P$ be a projective $A$-module of rank $n$. In this situation, when $k$ is an algebraically closed field, Murthy [18, Theorem 3.8] proved that $c_n(P) = 0$ is also sufficient, i.e. $c_n(P) = 0 \Rightarrow P \simeq A \oplus Q$. 

1.3. **Euler Class Groups**

Let us take a step back and look at the situation for Dedekind domains. Since the dimension of the ring is 1, Serre’s result implies that a finitely generated projective module \( P \) of rank \( t \) can be written as \( P \cong A^{t-1} \oplus Q \) for some rank 1 projective module \( Q \). Hence, we can restrict to the case where \( P \) has rank 1, in which case for Dedekind domains \( P \) is isomorphic to some nonzero ideal \( \mathfrak{a} \subseteq A \). Thus, the ideal \( \mathfrak{a} \) completely determines the structure of the module \( P \) and to check whether the rank 1 projective module \( P \) is free is same as checking whether the ideal \( \mathfrak{a} \) is principal.

Consider now the group defined as the set of non-zero ideals (which is actually a monoid) modulo the subset of principal ideals with ideal multiplication being the natural operation, the so-called “ideal class group”. Note that since every ideal is a product of powers of maximal ideals, this group can be expressed as the free abelian group generated by maximal ideals modulo the subgroup generated by principal ideals. Thus, one checks if the ideal class of \( \mathfrak{a} \) in the ideal class group is 0 and if so, one knows it is a principal ideal. Thus, the ideal class group provides **precise** obstructions for projective modules to split off free summands (in this case it means being free).

In particular for a smooth irreducible affine algebraic curve \( X \), this gives a precise obstruction for splitting. Note that in this case, the group \( \text{CH}_0(X) \) is precisely the ideal class group and the top Chern class is exactly the obstruction in this group. Murthy’s theorem then is a higher dimensional analogue of this situation when the base field is algebraically closed and says that the top Chern class is the precise obstruction to splitting.

Unfortunately, this is generally not true when the base field is not algebraically closed, i.e. \( c_{\mathfrak{a}}(P) = 0 \not\Rightarrow P \cong A \oplus Q \). A standard counterexample is that of the module of differentials \( \Omega_{A/R} \) where \( A \) is the coordinate ring of the even dimensional real sphere.

**1.3 Euler class groups**

Clearly then one would need a finer invariant to try and capture when a projective module of top rank splits off a free summand of rank 1. Let \( X = \text{Spec}(A) \) be
a smooth affine variety of dimension $n \geq 2$ over a field $k$ of characteristic 0 (in dimension 1, these are Dedekind domains). Then we can define a group called the Euler class group (a notion due to Nori) which is a generalisation of the ideal class group (refer section 2 for more details) as the free abelian group on maximal ideals with orientations modulo the subgroup generated by elements associated to oriented complete intersections of height $n$. This group was formally defined, initially for smooth, affine domains over fields of characteristic 0 and subsequently for any commutative, Noetherian ring containing $\mathbb{Q}$ in several papers by Bhatwadekar, Mandal and Raja Sridharan in 1990's (refer [6] for the most general definition and to [20] and [5] for how the definition evolved). One can associate to every projective module $P$ of rank $n$ and an orientation $\chi$, an element of the Euler class group (refer to section 1 for details) $e(P, \chi)$. From the definition, it is easy to show that $P \simeq A \oplus Q \Rightarrow e(P, \chi) = 0$. In [6], it is shown that the converse also holds, thus proving that this is indeed the finer invariant that was sought.

Let us turn back to the original problem of determining when the top Chern class is the precise obstruction to splitting. At the end of the previous section, we gave a counterexample when the vanishing of the top Chern class is not sufficient to conclude splitting. Note that the counterexample was in even dimension and interestingly, all known counterexamples have even dimension. This led people to believe that the vanishing of the top Chern class is sufficient when the dimension $n$ (which is same as the rank of the projective module) is odd.

Using the Euler class group, this was proved in the case $k = \mathbb{R}$, when the projective module is oriented (i.e. $\wedge^n(P) \simeq A$) in [5] and subsequently removing the orientedness condition in [2]. In fact in these papers, the authors completely characterised when $k = \mathbb{R}$ and $c_n(P) = 0$ implies $P \simeq A \oplus Q$, i.e. they prove the statement when $n$ is odd and give a series of topological conditions when $n$ is even under which they prove the statement. Further, they give counterexamples when none of the conditions holds true. The key tool in obtaining these results is a structure theorem for the Euler class group.
1.4 Main Theorems

In this thesis, we generalise the above mentioned to the case when the base field is a real closed field, that is a field which possesses the algebraic properties of \( \mathbb{R} \) (refer preliminaries for definition and details about these). Our main theorem is a structure theorem for the Euler class group as stated below:

**Theorem A.** Let \( R \) be a real closed field. Let \( X = \text{Spec}(A) \) be a smooth affine variety of dimension \( n \geq 2 \) over \( R \). Let \( X(R) \) denote the \( R \)-rational points of the variety. Let \( K \) denote the module \( \wedge^n(\Omega_A/R) \). Let \( P \) be a projective \( A \)-module of rank \( n \) and let \( \wedge^n(P) = L \). Let \( C_i, 1 \leq i \leq t \) be the closed and bounded semialgebraically connected semialgebraic components of \( X(R) \). Let \( L_i \) and \( K_i \) be the restriction of the semialgebraic line bundles corresponding to \( L \) and \( K \) respectively, to \( C_i \). Let \( L_i \simeq K_i \), for \( 1 \leq i \leq r \) and \( L_i \nsubseteq K_i \), for \( r + 1 \leq i \leq t \). Let \( x_i \in C_i \) and let \( M_i \) be the corresponding maximal ideal of \( R(A) \). Let \( \omega_i \) be a local \( R(L) \)-orientation of \( M_i \). Then,

\[
\bigoplus_{i=1}^r \mathbb{Z}e_i \bigoplus \bigoplus_{i=r+1}^t (\mathbb{Z}/2)e_i \xrightarrow{\sim} E(R(A), R(L))
\]

sending \( e_i \mapsto (M_i, \omega_i) \) is an isomorphism.

We note that in [2] (and before that in [5]), the proofs crucially involved using the topology of \( \mathbb{R} \), in particular the connectedness of intervals in \( \mathbb{R} \). Since such techniques are unavailable for real closed fields (in fact, open intervals in real closed fields other than \( \mathbb{R} \) are totally disconnected) this proof is completely algebraic. This gives reason to hope that when \( n \) is odd and \( c_n(P) = 0 \), then \( P \simeq A \oplus Q \) without any restriction on the base field.

As mentioned in the previous section, Theorem 1.4 allows us to deduce the following important corollary regarding splitting of projective modules, which generalises the results of [5] and [2].

**Theorem B.** Let \( R \) be a real closed field. Let \( X = \text{Spec}(A) \) be a smooth affine variety of dimension \( n \geq 2 \) over \( R \). Let \( X(R) \) denote the \( R \)-rational points of the variety. Let \( K \) denote the module \( \wedge^n(\Omega_A/R) \). Let \( P \) be a projective \( A \)-module of rank \( n \) and let \( \wedge^n(P) = L \). Assume that \( c_n(P) = 0 \) in \( CH_0(X) \). Then \( P \simeq A \oplus Q \) in the following cases:
1. $X(\mathbb{R})$ has no closed and bounded semialgebraically connected component.

2. For every closed and bounded semialgebraically connected component $W$ of $X(\mathbb{R})$, $L_W \not\cong K_W$ where $K_W$ and $L_W$ denote restriction of (induced) line bundles on $X(\mathbb{R})$ to $W$.

3. $n$ is odd.

Moreover, if $n$ is even and $L$ is a rank 1 projective $A$-module such that there exists a closed and bounded semialgebraically connected component $W$ of $X(\mathbb{R})$ with the property that $L_W \cong K_W$, then there exists a projective $A$-module $P$ of rank $n$ such that $P \oplus A \cong L \oplus A^{n-1} \oplus A$ (hence $c_n(P) = 0$) but $P$ does not have a free summand of rank 1.
Chapter 2

Preliminaries

2.1 The Euler Class Group and \( CH_0(X) \)

In this section, we introduce the Chow groups with particular emphasis on the group of zero cycles modulo rational equivalence, then give a brief survey of the theory of the Euler Class Group and finally draw a link between the two groups.

Let \( A \) be a smooth, affine domain of dimension \( n \) and let \( X = \text{Spec}(A) \). Consider the free abelian group \( Z_i(A) \) on the set

\[
\{ [p] | p \in \text{Spec}(A), \dim(A/p) = i \},
\]

the group of cycles of dimension \( i \). For an \( A \)-module \( M \) with \( \dim(M) \leq i \), let the cycle of \( \dim i \) associated to \( M \) be \( \sum_{\dim(A/p)=i} \text{length}_{A/p}(M|_p)|p| \).

Let \( I \) be an ideal of height \( n \) in \( A \), then \([I] \) denotes the cycle associated to \( A/I \) in \( Z_i(A) \). In the literature, this is referred to as \([A/I] \). However, as will be explained later, for our purposes, the notation \([I] \) is more convenient.

Note that for a prime \( q \) such that \( \dim(A/q) = i+1 \) and \( f \in A \setminus q \), either \( (q, f) \) is a proper ideal with \( \dim(A/(q, f)) = i \) or \( A = (q, f) \). In either case, we can associate an element of \( Z_i(A) \) to it since \( \dim(A/(q, f)) \leq i \). Notice that when \( A = (q, f) \), the associated element is 0. In either case, we denote this element by \([q, f] \) which is consistent with our notation for an ideal \( I \) above.

We define the \( i \)-th Chow group \( CH_i(A) \) as the quotient of \( Z_i(A) \) by the subgroup generated by \( \{ [(q, f)] | \dim(A/q) = i+1, f \in A \setminus q \} \). In the literature,
this group is also commonly referred to as $CH^{n-i}(A)$ as we have referred to it in section 1.2.

If we specialise to the case that $i = 0$, we see that $CH_0(A)$ is a quotient of the free abelian group on the maximal ideals by the subgroup generated by cycles of the type $[(q, f)]$ where $q$ is a prime ideal of height $n - 1$ and $f \notin q$. In this thesis, we will have occasion to use only this group and hence references to "the Chow group" are to this group. Further, we use both notations $CH_0(A)$ or $CH_0(X)$ to refer to this group.

We give another description which was the basis for Nori’s idea about the Euler class group [19, Theorem 5.4]. Consider the quotient of the free abelian group on the maximal ideals modulo the subgroup generated by cycles corresponding to the ideals of height $n$ which are generated by $n$ elements (complete intersection ideals), i.e.

$$ F \ < \ [m] \mid \frac{\mathbb{A}}{k} \text{ is a field} > $$

$$ < \ [(a_1, a_2, \ldots, a_n)] \mid \text{ht.}(a_1, a_2, \ldots, a_n) = n > $$

Then, Nori proved that this group is canonically isomorphic to $CH_0(X)$. Since we will be working with smooth, affine varieties, we will have occasion to use this description as well.

Let $I$ be an ideal of height $n$ in $A$, then $[I]$ was used to denote the cycle associated to $I$. By abuse of notation, we will also use the same notation to denote its image in $CH_0(X)$. In this thesis, we will really be using its image in $CH_0(X)$ rather than the cycle itself and hence there should be no confusion.

Now, given a projective module $P$ of rank $n$, let $\alpha : P \to A$ be a map such that the image $\alpha(P)$ is an ideal of height $n$. Such maps are called generic surjections and they exist by the Eisenbud-Evans theorem 2.4.4. If $\alpha$ and $\beta$ are two generic surjections, then it can be shown that $[\alpha(P)] = [\beta(P)]$ in $CH_0(A)$. This element is called the $n$-th Chern class $c_n(P)$ in $CH_0(A)$ (there are other more functorial ways of defining this element).

We move on to define the Euler class group. The definition we initially present will be the one we use in the thesis. This definition is specific to the case when the ring is a smooth affine domain. This is also how the Euler class group initially arose ([5], [20]) before it was generalised to any commutative Noetherian ring (we
will also present that definition later). More details for this part can be obtained either in [6] or [2] amongst other places.

**Definition 2.1.1.** Definition of $E(A, L)$ and $E_0(A, L)$

Let $A$ be a smooth affine domain of dimension $n \geq 2$ and let $L$ be a projective $A$-module of rank 1. Let $\mathcal{M}$ be a maximal ideal of $A$ of height $n$. Then, $\mathcal{M}/\mathcal{M}^2$ is generated by $n$ elements. An isomorphism $\omega_{\mathcal{M}} : L/\mathcal{M}L \xrightarrow{\sim} \Lambda^n(\mathcal{M}/\mathcal{M}^2)$ is called a local $L$-orientation of $\mathcal{M}$. Let $G$ be the free abelian group on the set of pairs $(\mathcal{M}, \omega_{\mathcal{M}})$ where $\mathcal{M}$ is a maximal ideal of height $n$ and $\omega_{\mathcal{M}}$ is a local $L$-orientation of $\mathcal{M}$.

Let $J = \cap_{i=1}^{k} \mathcal{M}_i$ be an intersection of finitely many maximal ideals of height $n$. Then, $J/J^2$ is generated by $n$ elements. An isomorphism $L/JL \xrightarrow{\sim} \Lambda^n(J/J^2)$ is called a local $L$-orientation of $J$. A local $L$-orientation of $J$ gives rise to local $\mathcal{M}_i$-orientations $\omega_{\mathcal{M}_i}, i = 1, 2, \ldots, k$. Then, we denote the element $\sum_{i=1}^{k} (\mathcal{M}_i, \omega_{\mathcal{M}_i})$ in $G$ as $(J, \omega_J)$.

A local $L$-orientation $\omega : L/JL \to \Lambda^n(J/J^2)$ is called a global $L$-orientation if there exists a surjection $\theta : L \oplus A^{n-1} \to J$, such that $\omega$ is the induced isomorphism $L/JL \xrightarrow{\alpha} \Lambda^n(L/JL \oplus (A/J)^{n-1}) \xrightarrow{\sim} \Lambda^n(J/J^2)$ where $\alpha(\bar{e}) = \bar{e}_1 \wedge \bar{e}_2 \wedge \ldots \wedge \bar{e}_n$ (and \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\} is a basis of $A^{n-1}$).

Let $H$ be the subgroup of $G$ generated by the set of pairs $(J, \omega_J)$, where $J$ is a finite intersection of maximal ideals of height $n$ and $\omega_J$ is a global $L$-orientation of $J$. The Euler class group of $A$ with respect to $L$ is $E(A, L) = G/H$. We write $E(A)$ for $E(A, A)$.

Further, let $G_0$ be the free abelian group on the set $(\mathcal{M})$ where $\mathcal{M}$ is a maximal ideal of $A$. Let $J = \cap_{i=1}^{k} \mathcal{M}_i$ be a finite intersection of maximal ideals. Let $(J)$ denote the element $\sum_{i}(\mathcal{M}_i)$ of $G_0$. Let $H_0$ be the subgroup of $G_0$ generated by elements of the type $(J)$, where $J$ is a finite intersection of maximal ideals such that there exists a surjection $\alpha : L \oplus A^{n-1} \to J$. Then, $E_0(A, L) := G_0/H_0$. From the definitions of $E(A, L)$ and $E_0(A, L)$, it is clear that there is a canonical surjection $E(A, L) \to E_0(A, L)$.

Now let $P$ be a projective $A$-module of rank $n$ such that $L \cong \Lambda^n(P)$ and let $\chi : L \xrightarrow{\sim} \Lambda^nP$ be an isomorphism. Let $\varphi : P \to J$ be a surjection where $J$ is a finite intersection of maximal ideals of height $n$. Therefore we obtain an induced
isomorphism $\varphi : P/JP \to J/J^2$. Let $\omega_J$ be the local $L$-orientation of $J$ given by $\wedge^n(\varphi) \circ \chi$. Let $e(P, \chi)$ be the image in $E(A, L)$ of the element $(J, \omega_J)$ of $G$. The assignment sending the pair $(P, \chi)$ to the element $e(P, \chi)$ of $E(A, L)$ can be shown to be well defined from [4, Section 4]. The \textit{Euler class} of $(P, \chi)$ is defined to be $e(P, \chi)$.

As mentioned earlier, there is also a more general definition for any commutative Noetherian ring of dimension $n$ containing $\mathbb{Q}$. This definition is more technical but allows more flexibility. We will need it to prove some key lemmas.

Let $A$ be a Noetherian ring containing $\mathbb{Q}$ with $\dim(A) = n \geq 2$. Let $L$ be a rank 1 projective $A$-module. We define the Euler Class group $E(A, L)$ of $A$ with respect to $L$ as follows:

Let $J \subset A$ be an ideal of height $n$ such that $J/J^2$ is generated by $n$ elements. Consider the set of surjections from $L/JL \oplus (A/J)^{n-1}$ to $J/J^2$. We define an equivalence relation on this set by declaring two surjections $\alpha$ and $\beta$ to be equivalent if there exists an automorphism $\sigma$ of $L/JL \oplus (A/J)^{n-1}$ with determinant 1 such that $\alpha \sigma = \beta$. We call such an equivalence class $[\alpha]$ a local $L$-orientation of $J$.

Note that since $\dim(A/J) = 0$, $E_{A/J}(L/JL \oplus (A/J)^{n-1}) = SL_{A/J}(L/JL \oplus (A/J)^{n-1})$ and hence, by [3, Proposition 4.1], the canonical map from $SL_A(L \oplus A^{n-1})$ to $SL_{A/J}(L/JL \oplus (A/J)^{n-1})$ is surjective. Hence if a surjection $\alpha$ from $L/JL \oplus (A/J)^{n-1}$ to $J/J^2$ can be lifted to a surjection $\theta : L \oplus A^{n-1} \to J$, then so can any surjection equivalent to $\alpha$.

A local $L$-orientation $\alpha$ of $J$ is called a global $L$-orientation of $J$ if the surjection $\alpha : L/JL \oplus (A/J)^{n-1} \to J/J^2$ can be lifted to a surjection $\theta : L \oplus A^{n-1} \to J$.

Let $\mathcal{M} \subset A$ be a maximal ideal of height $n$ and $\mathcal{N}$ be an $\mathcal{M}$-primary ideal such that $\mathcal{N}/\mathcal{N}^2$ is generated by $n$ elements. Let $\omega_{\mathcal{N}}$ be a local $L$-orientation of $\mathcal{N}$. Let $G$ be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal and $\omega_{\mathcal{N}}$ is a local $L$-orientation of $\mathcal{N}$.

Let $J = \bigcap \mathcal{N}_i$ be the intersection of finitely many ideals $\mathcal{N}_i$ where $\mathcal{N}_i$ is $\mathcal{M}_i$-primary ($\mathcal{M}_i \subset A$ are distinct maximal ideals of height $n$). Assume that $J/J^2$ is generated by $n$ elements. Let $\omega_J$ be a local $L$-orientation of $J$. Then, $\omega_J$ gives
rise, in a natural way, to a local $L$-orientation $\omega_{N_i}$ of $N_i$. We define $(J, \omega_J)$ to be the element \(\sum(N_i, \omega_{N_i})\) of $G$.

Let $H$ be the subgroup of $G$ generated by the set of pairs $(J, \omega_J)$, where $J$ is an ideal of height $n$ and $\omega_J$ is a global $L$-orientation of $J$. Then we define the Euler class group as $E(A, L) := G/H$.

Let $P$ be a projective module of rank $n$ with determinant $L$. Then, given an isomorphism $\chi : \wedge^n(L \oplus A^{n-1}) \cong \wedge^n(P)$, we can associate an element $e(P, \chi)$ of the group $E(A, L)$ to this pair, called the Euler class of $(P, \chi)$.

**Remark 2.1.2.** [6, Remark 4.7] We note that both the definitions of $E(A, L)$ and $e(P, \chi)$ coincide when the commutative Noetherian ring $A$ happens to be a smooth, affine domain over a field of characteristic 0.

The reason for the importance of this element (and hence the group) is the following crucial theorem:

**Theorem 2.1.3.** [6, Corollary 4.4] Let $A$ be a commutative, Noetherian ring of dimension $n \geq 2$ containing the field $\mathbb{Q}$. Let $L$ be a projective $A$-module of rank 1 and $P$ be a projective $A$-module of rank $n$ with $L \cong \wedge^n(P)$. Let $\chi : L \cong \wedge^n P$ be an isomorphism. Let $J \subset A$ be an ideal of height $n$ and $\omega_J$ be a local $L$-orientation of $J$. Then,

1. Suppose that $(J, \omega_J)$ is zero in $E(A, L)$. Then there exists a surjection $\alpha : L \oplus A^{n-1} \to J$ such that $\omega_J$ is induced by $\alpha$ (in other words, $\omega_J$ is a global $L$-orientation).

2. $P \cong Q \oplus A$ for some projective $A$-module $Q$ of rank $n - 1$ if and only if $e(P, \chi) = 0$ in $E(A, L)$.

The reason for calling it the Euler class is that in topology, one associates a class (again called the Euler class) to any oriented vector bundle of rank $n$ in the $n$-th homology group of the base space (which is also assumed oriented) with $\mathbb{Z}$ coefficients. When $n$ also happens to be the dimension of the base space, the Euler class vanishes if and only if the vector bundle splits off a free summand of rank 1, exactly as in the theorem above.

**Remark 2.1.4.** Note that if $\omega_0$ and $\omega_1$ are two local orientations of a reduced ideal $J$, then $\omega_0 = \lambda \omega_1$ where $\lambda \in (A/J)^\ast$. 

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2.1. THE EULER CLASS GROUP AND $CH_0(X)$ 15
The next lemma give us some tools to make computations in the Euler class group.

**Lemma 2.1.5.** [6, Lemma 5.4] Let $A$ be a Noetherian ring of dimension $n \geq 2$ containing $\mathbb{Q}$. Let $J \subset A$ be an ideal of height $n$ and $\omega_J$ be a local $L$-orientation of $J$. Let $\pi \in A/J$ be a unit. Then $(J, \omega_J) = (J, \overline{\omega}_J)$ in $E(A, L)$.

We now connect $\text{CH}_0(X)$ with the Euler class group when $X = \text{Spec}(A)$ is a smooth, affine variety over a field of characteristic 0. This link between the $\text{CH}_0(X)$ and $E(A, L)$ will be crucially used in the chapter 6 in order to compute $E(A, L)$.

**Remark 2.1.6.** Recall that if $I$ is an ideal in $A$ of height $n$ such that $I/I^2$ is generated by $n$ elements, then $[I]$ denotes the cycle associated to $A/I$ in $\text{CH}_0(X)$ and $(I)$ denotes the element of $E_0(A, L)$ associated to $I$.

From the above definitions, it is clear that there exists a natural surjection $\Theta_L : E(A, L) \to \text{CH}_0(X)$ with the property that if $P$ is a projective $A$-module of rank $n$ with an isomorphism $\chi : \wedge^n(P) \simeq L$, then $\Theta_L(e(P, \chi)) = c_n(P)$, where $c_n(P)$ denotes the $n$th Chern class of $P$ (which is an element of $\text{CH}_0(X)$). Therefore, in view of (2.1.3), in order to conclude that $P \simeq A\oplus Q$ given $c_n(P) = 0$, it is enough to prove that $e(P, \chi) = 0$ for some $\chi$, and hence computation of the Euler class group $E(A, L)$ is crucial.

### 2.2 Real closed fields and semialgebraic sets

In this subsection, we discuss the theory of real closed fields and the topological notions related to them. This section is somewhat crucial in understanding the similarities and differences between working with an arbitrary real closed fields $\mathbb{R}$ and $\mathbb{R}$. More details can be found in [10].

**Definition 2.2.1.** A field $\mathbb{R}$ is said to be real if it can be ordered in a way such that addition and multiplication are compatible with the ordering. An equivalent definition is that $\sum_{i=1}^n a_i = 0 \Rightarrow a_i = 0 \forall i$. A real closed field is a real field which has no algebraic extensions which are real, equivalently attaching a root of $-1$ makes it algebraically closed.
2.2. REAL CLOSED FIELDS AND SEMIALGEBRAIC SETS  

Note that being ordered, real fields have characteristic 0. Real closed fields come with a natural topology based on intervals like in the case of \( \mathbb{R} \). We can extend this topology to \( \mathbb{R}^l \) (product topology). We call this topology the Euclidean topology. Note that this topology comes from a “metric” taking values in \( \mathbb{R} \), namely \( d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{l} (x_i - y_i)^2} \) where \( \mathbf{x} = (x_1, x_2, \ldots, x_l) \) and \( \mathbf{y} = (y_1, y_2, \ldots, y_l) \).

Note that for a subset \( V \subset \mathbb{R}^l \), there is an inherited Euclidean topology and associated “metric”. Thus, one can talk of open, closed and bounded sets in \( V \).

The above definition essentially abstracts out the algebraic properties of \( \mathbb{R} \). Clearly, \( \mathbb{R} \) is the primary example of a real closed field. We mention some other examples of real fields and real closed fields. Note that from the definition, it is clear that any subfield of a real field is again a real field. Hence, all subfields of \( \mathbb{R} \) are real fields. These fields are called Archimedean real fields (since they satisfy the Archimedean property i.e. given \( a, b > 0 \), there exists \( n \in \mathbb{N} \) such that \( na > b \)). In particular, \( \mathbb{Q} \) and extensions of \( \mathbb{Q} \) obtained by attaching roots of polynomials with real roots are real fields. This leads to the smallest real closed field, namely, the field obtained by considering all elements of \( \mathbb{R} \) which satisfy a polynomial over \( \mathbb{Q} \) (the real algebraic closure of \( \mathbb{Q} \)). This can also be viewed as \( \overline{\mathbb{Q}} \cap \mathbb{R} \) where \( \overline{\mathbb{Q}} \) is the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \).

We mention another example. Let \( \mathbb{F} \) be a field and consider the new field

\[
P(\mathbb{F}) = \left\{ \sum_{k \geq m, k \in \mathbb{Z}} a_k X^{k/n} \mid m \in \mathbb{Z}, n \in \mathbb{N}, a_k \in \mathbb{F} \right\} = \bigcup_{k=1}^{\infty} \mathbb{F}((X^{1/n}))\]

the so-called Puiseux series over \( \mathbb{F} \). Puiseux (and possibly before him Sir Isaac Newton) proved that when \( \mathbb{F} \) is an algebraically closed field, the above field is actually an algebraically closed field. It is then easy to see that when \( \mathbb{F} \) is real closed, so is \( P(\mathbb{F}) \). Note that since the unique order on this field is given by \( X < a \) for all positive real numbers \( a \), this field does not satisfy the Archimedean property.

We now note a crucial difference between arbitrary real closed fields and \( \mathbb{R} \). Under this Euclidean topology, intervals in the field are not connected, except in the case of \( \mathbb{R} \) ([10, Section 2.4]).
We give two examples:

1. For any Archimedean real closed field $\mathbb{R}$ other than $\mathbb{R}$, i.e. real closed fields properly contained in $\mathbb{R}$, any interval, say $(0, 1) \cap \mathbb{R}$ we can choose a real number $a \notin \mathbb{R}$ such that $0 < a < 1$ in $\mathbb{R}$ and then consider $(0, a) \cap \mathbb{R}$ and $(a, 1) \cap \mathbb{R}$ which are disjoint open sets of $(0, 1) \cap \mathbb{R}$ which span it.

2. In the field of Puiseux series $P(\mathbb{R})$, the set

$$\{ f \in P(\mathbb{R}) | \exists r \in \mathbb{R}, r > 0, f > r \}$$

is both open and closed.

To overcome this difficulty of being in a world where things are disconnected, we introduce a slightly different notion of connectedness called semialgebraic connectedness, which we will make crucial use of in subsequent proofs. In fact, the notion of semialgebraicity is defined below for sets and functions as well.

**Definition 2.2.2.**

- A subset $V$ of $\mathbb{R}^l$ is called a basic semialgebraic set if $V$ is of the form

$$\{ x \in \mathbb{R}^l | f_i(x) = 0, g_j(x) > 0, 1 \leq i \leq r, 1 \leq j \leq s \},$$

where $f_i(x), g_j(x) \in \mathbb{R}[X_1, X_2, \ldots, X_l]$. A subset $W$ of $\mathbb{R}^l$ is called a semialgebraic set if $W$ is a finite union of basic semialgebraic sets. Note that in particular, intervals are (basic) semialgebraic sets.

- A semialgebraic subset $W$ of $\mathbb{R}^l$ is semialgebraically connected if for every pair of disjoint, closed, semialgebraic subsets $F_1$ and $F_2$ of $W$ satisfying $F_1 \cup F_2 = W$, either $F_1 = W$ or $F_2 = W$.

- A map between two semialgebraic sets $f : A \rightarrow B$ is said to be semialgebraic if its graph is a semialgebraic set.

- $C^k(U, B)$ is defined to be the class of functions from an open semialgebraic set $U$ to a semialgebraic set $B$ for which all partial derivatives up to order $k$ exist and are continuous where $k = 0, 1, \ldots, \infty$. 
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- $S^k(U, B)$ is defined to be the class of semialgebraic functions from an open semialgebraic set $U$ to a semialgebraic set $B$ of class $C^k$ where $k = 0, 1, \ldots, \infty$.

- A semialgebraic function which is of class $C^\infty$ is called a Nash function.

- A semialgebraic path in a semialgebraic set $V$ is the image of a continuous, semialgebraic map $f : [0, 1] \to V$.

Now we quote a result, the proof of which can be found in [10, Theorem 2.4.4].

**Theorem 2.2.3.** Every semialgebraic subset $W$ of $R^l$ is the disjoint union of a finite number of semialgebraically connected semialgebraic subsets $W_1, W_2, \ldots, W_s$ which are closed in $W$. The $W_1, W_2, \ldots, W_s$ are called the **semialgebraically connected semialgebraic components** of $W$. By abuse of notation, we shall refer to them simply as components of $W$.

To convince the reader that these concepts only generalise already known things for $R$, we mention that when the field is $R$, the semialgebraically connected semialgebraic components are same as the connected components by [10, Theorem 2.4.5].

Note that two points $P$ and $Q$ of a semialgebraic set $W$ lie in the same component of $W$ if and only if they can be joined by a semialgebraic path in $W$ as stated in [10, Proposition 2.5.13].

To increase familiarity with these concepts, we give a few examples and pictures. It is clear that algebraic sets are semialgebraic sets and polynomial functions are semialgebraic functions. Further, sets which one regularly comes across are semialgebraic, for example the unit circle, unit square, solid sphere, solid torus. Further, taking roots (when defined) is also semialgebraic. Thus, functions like $\sqrt{x}$ (on its domain of definition), $\sqrt[3]{x}$, etc. are semialgebraic.

**Remark 2.2.4.** $R^n \setminus \{0\}$ is semialgebraically connected for all $n \geq 2$. However, for $n = 1$, i.e. $R \setminus 0$ has 2 components, namely $\{x > 0\}$ and $\{x < 0\}$. Hence, for any semialgebraically connected set $W$, $W \times (R \setminus 0)$ also has 2 components, namely $W \times \{x > 0\}$ and $W \times \{x < 0\}$. 
Infinite Zigzag: This is not semialgebraic.
Note however that a finite zigzag is indeed semialgebraic.

Exponential function: This is the set \((x, y) | y = e^x\).
This is not a semialgebraic set and hence \(e^x\) is not a semialgebraic function.

\[(x, y) | y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\): This is semialgebraic.
Note how similar the graphs in the above two examples look.

PacMan: Semialgebraic sets can also be fun!!!
Note that this set is not algebraic.

We now state the semialgebraic version of the implicit function theorem as stated in [10, Corollary 2.9.8] which we will require in chapter 4.
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**Theorem 2.2.5.** Let \((x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^{l-n}\) and let \(f_1, f_2, \ldots, f_{l-n}\) be semialgebraic functions of class \(C^k\) on an open neighbourhood of \((x^0, y^0)\), such that \(f_j(x^0, y^0) = 0\) for \(j = 1, 2, \ldots, l - n\) and the matrix

\[
\left[ \frac{\partial f_j}{\partial y_i}(x^0, y^0) \right]
\]

is invertible. Then there exist open semialgebraic neighbourhoods \(U \subset \mathbb{R}^n\) of \(x^0\) and \(V \subset \mathbb{R}^{l-n}\) of \(y^0\) and a semialgebraic mapping \(\phi \in \mathcal{S}^k(U, V)\) such that \(\phi(x^0) = y^0\) and

\[f_1(x, y) = \ldots = f_{l-n}(x, y) = 0 \iff y = \phi(x)\]

for every \((x, y) \in U \times V\).

We now define a semialgebraic vector bundle and what it means for two such to be semialgebraically isomorphic.

**Definition 2.2.6.** [10, 12.7.1] Let \(M \subset \mathbb{R}^n\) be a semialgebraic set and \(E\) be a topological space with a continuous map \(p : E \to M\) such that each fibre \(p^{-1}(m)\) is an \(n\)-dimensional vector space over \(\mathbb{R}\). Suppose there exists a finite cover of \(M\) by semialgebraic, open subsets \(\{U_i\}_{i \in I}\) and maps \(\phi_i : U_i \times \mathbb{R}^n \overset{\sim}{\to} p^{-1}(U_i)\) which are homeomorphic and induce vector space isomorphisms at each fibre. Further suppose that \(\phi_i^{-1} \circ \phi_j|_{(U_i \cap U_j) \times \mathbb{R}^n}\) is semialgebraic for all pairs \((i, j) \in I \times I\). Then this is an \(n\)-dimensional vector bundle \((E, p, M)\) over \(\mathbb{R}\) with a semialgebraic atlas. Two atlases are called equivalent if the union of the covers again forms a semialgebraic atlas. This is an equivalence relation. Then a semialgebraic vector bundle of rank \(n\) is an \(n\)-dimensional bundle \((E, p, M)\) over \(\mathbb{R}\) with a fixed equivalence class of semialgebraic atlases.

Notice that though \(E\) was a priori not assumed to be semialgebraic, the above definition implies that it is a semialgebraic set.

**Definition 2.2.7.** Contd. A semialgebraic map \(\phi\) of semialgebraic bundles \(\mathcal{E}_1 = (E_1, p_1, M, (U_i, \phi_i)_{i \in I})\) and \(\mathcal{E}_2 = (E_2, p_2, M, (U'_j, \phi'_j)_{j \in J})\) over the same base space \(M\) is a map between the top spaces which induces vector space homomorphisms
on all fibres, induces a commutative diagram

\[ \begin{array}{ccc}
    E_1 & \xrightarrow{\phi} & E_2 \\
    \downarrow{p_1} & & \downarrow{p_2} \\
    M & \xleftarrow{j} & 
\end{array} \]

and such that \( \phi_j^{-1} \circ \phi \circ \phi_i|_{(U_i \cap U_j) \times \mathbb{R}^n} \) are continuous and semialgebraic for all \((i, j) \in I \times J\).

A section of a semialgebraic vector bundle is a continuous, semialgebraic map \( M \to E \) such that \( p \circ s = id_M \).

By usual abuse of notation, several times we will write only \( E \) rather than the triple \((E, p, M)\). Further, in the situations we are in, the fixed equivalence class of atlases is implicit and will not be written.

Further, we will be considering only semialgebraic maps of semialgebraic vector bundles and hence we will refer to them only as maps or morphisms. If \( E_1 \) and \( E_2 \) are two semialgebraic line bundles, we will denote \( E_1 \simeq E_2 \) to mean that \( E_1 \) and \( E_2 \) are semialgebraically isomorphic, meaning there exist semialgebraic maps both ways which compose to the respective identity.

Note that there is always a canonical section, the 0-section for any vector bundle which is also semialgebraic and continuous, and hence is a section as defined above. Further, if for any vector bundle, there is a nowhere vanishing section, i.e. a section whose image does not intersect the image of the 0-section at any point, then the bundle splits off a free summand of rank 1. In our context, for a semialgebraic bundle \( E \), if there is a nowhere vanishing semialgebraic section, then we can write \( E \simeq E_1 \oplus E' \) where by \( E_1 \) we represent the semialgebraically trivial line bundle.

With this background, we prove a result for later use.

**Lemma 2.2.8.** [7, Lemma 3.10] Let \( W \) be a semialgebraically connected semialgebraic set and let \( \pi : E \to W \) be a semialgebraic line bundle. Then, \( E^* = E \setminus \{\text{zero section}\} \) has 2 components if and only if \( E \) is a semialgebraically trivial line bundle.
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Proof. If $E$ is semialgebraically trivial, (2.2.4) says that $E^* = E \setminus \{\text{zero section}\}$ has two semialgebraically connected semialgebraic components. We now prove the other part.

In view of [10, Lemma 12.7.3], it is easy to see that every semialgebraic vector bundle comes equipped with a continuous, semialgebraic map $E \oplus E \to R$ (called a Riemannian metric) such that restricted to each fibre, it is an inner product. Let $\theta$ be the composite map $E \xrightarrow{\Delta} E \oplus E \to R$ induced through the diagonal. Then, $E^* = \theta^{-1}(0, \infty)$. Let $E^* = U_1 \sqcup U_2, U_1 \neq \emptyset, U_2 \neq \emptyset$, where $U_1$ and $U_2$ are semialgebraic open subsets.

Step 1 : We shall prove that for every fibre the intersection $E_x$ with $U_1$ and $U_2$ is non-empty. Let $V_1 = \{ x \in W : U_1 \cap E_x \neq \emptyset, U_2 \cap E_x \neq \emptyset \} = \pi(U_1) \cap \pi(U_2)$

$V_2 = \{ x \in W : U_1 \cap E_x = \emptyset \} = \{ x \in W : E^*_x \subseteq U_2 \} = \pi(U_2) \cap (\pi(U_1))^c$ and

$V_3 = \{ x \in W : U_2 \cap E_x = \emptyset \} = \{ x \in X : E^*_x \subseteq U_1 \} = \pi(U_1) \cap (\pi(U_2))^c$.

Since a line bundle is locally trivial, all the three sets are clearly open in $W$. Since $\pi$ is semialgebraic, these sets are also semialgebraic. Again, they are clearly disjoint and span $X$. Since $X$ is semialgebraically connected, $W = V_i$ for some $i$.

If $W = V_2$, then $E^*_x \subseteq U_2 \forall x \in X \Rightarrow E^* = U_2 \Rightarrow U_1 = \emptyset$ which is a contradiction. Hence, $W \neq V_2$. Similarly, $W \neq V_3$. Hence, $W = V_1$. Therefore, given $x \in W$, $U_1 \cap E_x \neq \emptyset$ and $U_2 \cap E_x \neq \emptyset$. But $E^*_x$ has exactly two semialgebraically connected components. Therefore, they are exactly $U_1 \cap E_x$ and $U_2 \cap E_x$.

Step 2 : We will now define a map $\psi : W \to E^*$. Let $x \in W$ and restrict $\theta$ to $E_x$. This is the square of a norm on the 1-dimensional vector space over $R$. Therefore, there exist two elements of $E_x$ such that their evaluation under $\theta$ is 1. Further, if one of them is $t$, then the other is $-t$. From step 1, exactly one of these elements lies in $U_1$. In other words, $\theta^{-1}(1) \cap E_x \cap U_1$ is a singleton set.

Define the map $\psi : W \to E^*$ sending $x$ to the element of $\theta^{-1}(1) \cap E_x \cap U_1$. It is clear that $p \circ \psi = \text{id}_W$.

Step 3 : We will now show that $\psi$ is continuous and semialgebraic, i.e. $\psi$ is a section. We first note that there is a finite, open, semialgebraically connected, semialgebraic covering of our base space $W$ such that the bundle is trivial on them. Then, it is enough to prove that for each such open set $U$, $\psi|_U$ is continuous and semialgebraic.

Let $U$ be one of the open sets in the cover and let $\alpha$ be a local trivialization
of $\mathcal{E}$ over $U$, i.e.

$$\begin{array}{ccc}
U \times \mathbb{R} & \xrightarrow{\alpha} & \pi^{-1}(U) \\
\downarrow{p} & & \downarrow{\pi} \\
U & \xrightarrow{\pi^{-1}} & \mathbb{R}^+
\end{array}$$

Now, note that $\pi^{-1}(U) \cap \mathcal{E}^*$ has exactly two semialgebraically connected components, $\pi^{-1}(U) \cap \mathcal{U}_1$ and $\pi^{-1}(U) \cap \mathcal{U}_2$ which correspond to $U \times \mathbb{R}^+$ and $U \times \mathbb{R}^-$ through $\alpha$. W.l.o.g. we can always assume that $U \times \mathbb{R}^+ \supseteq \pi^{-1}(U) \cap \mathcal{U}_1$ (else we choose $-\alpha$). Then, it is enough to prove the composite

$$U \xrightarrow{\psi} \pi^{-1}(U) \cap \mathcal{U}_1 \xrightarrow{\alpha^{-1}} U \times \mathbb{R}^+$$

is continuous and semialgebraic. Let $w \in U$. Consider $\theta(\alpha(w, 1)) = r(w) > 0$. Then $r$ is a continuous, semialgebraic function on $U$. Therefore, we get

$$\frac{1}{r(w)} \theta(\alpha(w, 1)) = 1 \Rightarrow \theta(\frac{1}{\sqrt{r(w)}} \alpha(w, 1)) = 1 \Rightarrow \theta(\alpha(w, \frac{1}{\sqrt{r(w)}})) = 1.$$ 

$$\Rightarrow \alpha(w, \frac{1}{\sqrt{r(w)}}) \in \theta^{-1}(1) \cap \mathcal{E}_x \cap \mathcal{U}_1$$

$$\Rightarrow \psi(w) = \alpha(w, \frac{1}{\sqrt{r(w)}})$$

$$\Rightarrow \alpha^{-1}(\psi(w)) = (w, \frac{1}{\sqrt{r(w)}})$$

Hence, we are reduced to showing that $w \mapsto (w, \frac{1}{\sqrt{r(w)}})$ is continuous and semialgebraic. But this is true since $r$ was a continuous, non-zero, semialgebraic function. Hence, we are done.

We are now done with the exposition and results required for semialgebraic topology in this thesis. We set up some notations which we will use in the remaining part of the thesis.

A maximal ideal $\mathcal{M}$ of a ring $A$ is called a real maximal ideal if $A/\mathcal{M}$ is a real field. Note that if $A$ is an affine algebra over a real closed field $\mathbb{R}$, then every maximal ideal has residue field either $\mathbb{R}$ or its algebraic closure $\bar{\mathbb{R}}$. The
R-rational points are real maximal ideals while we refer to R-rational points as non-real maximal ideals (in some places in the literature they are referred to as complex maximal ideals).

Let $A$ be an affine algebra over a real closed field $R$ and $X = \text{Spec}(A)$. Then we denote by $R(A)$, the ring obtained by inverting all elements which do not belong to any real maximal ideal. This is same as the localisation $S^{-1}A$ where

$$S = \{1 + \sum_{i=1}^{n} f_i^2 | f_i \in A\}.$$ Note that maximal ideals of $R(A)$ are in one-to-one correspondence with real maximal ideals of $A$. We denote by $X(R)$ the set of all $R$-rational points of $X$. In this thesis, very often we do not distinguish between $R$-rational points of $X$ and the corresponding maximal ideals of $R(A)$. For a module $M$ over $A$, we denote the $R(A)$-module $M \otimes_A R(A)$ by $R(M)$. We remark that in the literature, the ring $R(A)$ is sometimes referred to as $R(X)$.

Note that if $A$ is an affine algebra over a real closed field $R$, then elements of $R(A)$ act as functions on $X(R)$ taking values in $R$ (canonically). There is a natural map $\text{sign} : R^* \to \{\pm 1\}$; namely $\text{sign}(\lambda) = 1$ if $\lambda > 0$ and $\text{sign}(\lambda) = -1$ if $\lambda < 0$. For a function $f$ taking values in $R^*$, we can then talk about $\text{sign}(f(P))$ for any point $P$ in the domain. In this sense, we use $\text{sign}(f(P))$ where $P \in X(R)$ and $f \in R(A)$ does not belong to the maximal ideal of $R(A)$ corresponding to $P$.

With the above set-up, we state a very important theorem about the Euler class group $\text{E}(R(A), R(L))$ which will be used crucially in chapter 6.

**Lemma 2.2.9.** [2, Lemma 4.2] Let $A$ be a smooth affine domain over $R$. Let $L$ be a projective $A$-module of rank 1. Let $M$ be a maximal ideal of $R(A)$ and $\omega_M$ be a local $L$-orientation of $M$. Then $(M, \omega_M) + (M, -\omega_M) = 0$ in $\text{E}(R(A), R(L))$. As a consequence, $\text{E}(R(A), R(L))$ is a vector space over $\mathbb{Z}/2$. Moreover, if $\tilde{\omega}_M$ is another local $L$-orientation of $M$ then either $(M, \tilde{\omega}_M) = (M, \omega_M)$ or $(M, \tilde{\omega}_M) = (M, -\omega_M)$ in $\text{E}(R(A), R(L))$.

### 2.3 Elementary Paths

We start by quoting a few results and setting up notations necessary for defining an elementary path.
Proposition 2.3.1. [7, Propn. 3.3] Let $R$ be a real closed field and let $B$ be an affine algebra over $R$. Let $E$ be a projective module of rank 1 over $R(B)$ generated by $\{e_1, e_2, \ldots, e_n\}$. Then, $\sum_{i=1}^n e_i \otimes e_i$ generates $E \otimes_{R(B)} E \cong R(B)$. Thus, the group of rank one projective $R(B)$-modules is 2-torsion.

Similarly, if $B$ is the coordinate ring of an affine variety over $R$, and $I$ is an invertible ideal of $R(B)$, such that $I = (a_1, a_2, \ldots, a_n)$ then $I^2 = (\sum_{i=1}^n a_i^2)$.

In particular, if $\dim(B) = 1$, then for any regular maximal ideal $m$ of $R(B)$, if $m = (a_1, a_2, \ldots, a_n)$, then $m^2 = (\sum_{i=1}^n a_i^2)$.

Proof. Note that the second and hence, third statement is implied by the first, because under the hypothesis of the second statement, $I$ is actually a rank 1 projective module.

We first note that if $R$ is a ring and $m$ is a maximal ideal of $R$ such that $R/m$ is a real field, then $\sum_{i=1}^n a_i^2 \in m$ implies that $\sum_{i=1}^n (a_i)^2 = 0$ in $R/m$ which means $a_i = 0 \forall i \Rightarrow a_i \in m \forall i$. Thus, $\sum_{i=1}^n a_i^2 \in m \Rightarrow a_i \in m \forall i$.

Let $(e_1, e_2, \ldots, e_n)$ be a set of generators for $E$. We claim that $e = \sum_{i=1}^n e_i \otimes e_i$ generates $E \otimes_{R(B)} E$. To check this, it is enough to check it in every localization at a maximal ideal. All maximal ideals are real. Let $m$ be one such. Then $E_m$ is a free module of rank 1, hence generated by $e_i$ for some $i, 1 \leq i \leq n$. Without loss of generality, we assume that it is generated by $e_1$. Then, $E_m \otimes_{R(B)} m E_m$ is generated by $e_1 \otimes e_1$. Further, $e_i = a_i e_1$, $a_i \in R(B)m$ and $a_1 = 1$. Then, $e = \sum_{i=1}^n e_i \otimes e_i = (\sum_{i=1}^n a_i^2)(e_1 \otimes e_1)$. Since $a_1 = 1$ and the residue field of the local ring is $R$, $\sum_{i=1}^n a_i^2$ is a unit in $R(B)m$. Thus, $E_m \otimes_{R(B)} m E_m$ is generated by $e = \sum_{i=1}^n e_i \otimes e_i$. Thus, we see that $E \otimes_{R(B)} E \cong R(B)$. \qed

We set up some further notations. Let $Z = \text{Spec}(C)$ be a smooth affine curve over a real closed field $R$. Let $\tilde{Z}$ be its smooth projectivisation. Then, we have a natural injection $Z \hookrightarrow \tilde{Z}$. Note that $Z$ is an open subset of $\tilde{Z}$. Hence, stalks will be isomorphic, and hence the local rings $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{\tilde{Z},z}$ will be same for points $z \in Z$. Hence, real points of $Z$ continue to be real points of $\tilde{Z}$, i.e. $Z(R) \hookrightarrow \tilde{Z}(R)$.

Since $R$ is not algebraically closed, all real points of $\tilde{Z}$ are actually contained in an affine, open subset of $\tilde{Z}$. Let this be $Z' = \text{Spec}(C')$. Then, consider $Z \cap Z' = \tilde{Z}$ which is affine. Let $\tilde{C}$ be the coordinate ring of $\tilde{Z}$. Note that since $Z'(R) = \tilde{Z}(R)$, we have $\tilde{Z}(R) = Z(R)$.
2.3. **ELEMENTARY PATHS**

Let $K(\tilde{Z})$ be the function field of $\tilde{Z}$. Then, $\mathcal{O}_{\tilde{Z},z} \hookrightarrow K(\tilde{Z})$ and hence,

$$R(C') = \cap_{z \in \tilde{Z}'(\mathbb{R})} \mathcal{O}_{\tilde{Z},z}, \quad R(C) = \cap_{z \in \tilde{Z}(\mathbb{R})} \mathcal{O}_{\tilde{Z},z} = \cap_{z \in \tilde{Z}(\mathbb{R})} \mathcal{O}_{\tilde{Z},z} = R(\tilde{C}).$$

Thus, $R(C') \hookrightarrow R(\tilde{C}) = R(C)$. Moreover, since $R(C')$ is a Dedekind domain, birational to $R(C)$ and Pic$(R(C'))$ is two-torsion, $R(C)$ is a localisation of $R(C')$.

Note that $\tilde{Z}$ is a smooth, complete curve and $Z'$ is an affine open subset containing all its real points. With this notation in mind, we quote some theorems and statements from [16] which are at the heart of the technical parts of this thesis. We note that some of these statements while not mentioned in the form we are using, are derivable from the results and proofs in [16].

**Theorem 2.3.2.** [16, Theorem 5.2] $\Omega_{R(C')/R}$ is a free module of rank 1 over $R(C')$.

Fix a generator of $\Omega_{R(C')/R}$, say $\chi$, which is regarded as a global "orientation". This continues to be a generator for $\Omega_{R(\tilde{C})/R}$. With this notation in mind, we obtain the following :

**Theorem 2.3.3.** [16, 4.5a-6.1-6.2] Given an ordered pair of distinct points $P, Q$ in the same component of $Z'(\mathbb{R})$, there is a function $f_{P,Q} \in R(C')$ with the following properties :

1. $(f_{P,Q}) = m_P \cap m_Q$ in $R(C')$.
2. if $df_{P,Q} = g \chi$, then it has opposite orientations at both points, i.e.
   $\text{sign}(g(P)) = -1, \text{sign}(g(Q)) = 1$.
3. $f_{P,Q}$ is positive at all points outside the component containing $P$ and $Q$.

**Remark 2.3.4.** We make a few observations about $f_{P,Q}$.

1. Note that the function $f_{P,Q}$ is not symmetric in $\{P, Q\}$, i.e. $f_{P,Q}$ is not same as $f_{Q,P}$ (due to condition 2 above) and hence in the hypothesis, we choose an "ordered pair of points". However, the above theorem implies the existence of a function $f_{Q,P}$ and one can ask if there is any way of obtaining such a function from $f_{P,Q}$. Indeed, there is!!!

In [16], one of the key theorems is that for any choice of signs on the components, one can obtain a function (i.e. an element of $R(C')$) satisfying...
that choice. Indeed, such an element is forced to be a unit of $\mathbf{R}(C')$. Then, we can choose a unit $u \in \mathbf{R}(C')$ which takes negative values on all components other than the one containing $P$ and $Q$, and takes positive values on the component containing $P$ and $Q$. Then, the element $-uf_{P,Q}$ satisfies all the properties above to be a candidate for $f_{Q,P}$.

2. The above theorem talks about existence of a function satisfying the properties 1, 2 and 3 but does not say anything about uniqueness. Note that if there are two functions $f_{P,Q}$ and $f'_{P,Q}$ satisfying the above properties, then $f_{P,Q} = uf_{P,Q}$ where $u \in \mathbf{R}(C')^*$ is such that $u(T) > 0 \forall T \in Z'(\mathbf{R})$ (Artin’s Theorem then says that $u$ is a sum of squares of rational functions).

The function $f_{P,Q}$ then defines an open subset

$$[P, Q[ = \{ T \in Z'(\mathbf{R}) | f_{P,Q}(T) < 0 \}$$

of the component containing $P$ and $Q$. Due to the uniqueness part of the above remark, it is well-defined. Then

$$[P, Q] = \{ T \in Z'(\mathbf{R}) | f_{P,Q}(T) \leq 0 \} = [P, Q[ \cup \{ P \} \cup \{ Q \}$$

is a closed set and is actually the closure of $[P, Q[$ in the Euclidean topology on $Z'(\mathbf{R})$. By definition, these sets are semialgebraic subsets of $Z'(\mathbf{R})$. It can be deduced from [16] that they are semialgebraically connected.

Further, thanks to the description in the above remark of $f_{Q,P}$, we get that the component $W$ containing $P$ and $Q$ can be written as a disjoint union

$$W = \{ P \} \cup \{ Q \} \cup [P, Q[ \cup ]Q, P[.$$  

From this, we can deduce that if $P, R, S$ are distinct points of $W$, then either $[P, R] \nsubseteq [P, S]$ or $[P, S] \nsubseteq [P, R]$. 

Now, let $R, S \in [P, Q[$ be distinct points. Then, we can define a total order on $[P, Q[$ by defining $R < S$ if $[P, R] \subseteq [P, S]$. This order naturally extends to $[P, Q]$ by letting $P < R < Q$ for all $R \in [P, Q[.$

In fact Knebusch proves that there is a bijective semialgebraic order preserving homeomorphism between $[P, Q[$ and the open interval $]0, 1[ \subset \mathbf{R}$. Hence, we refer
2.3. ELEMENTARY PATHS

to sets \([P, Q]\) as open intervals and \([P, Q]\) as closed intervals of \(Z'(\mathbb{R})\). We refer to [16, Section 6] and [17] for more details.

Note that since \(Z(\mathbb{R}) \hookrightarrow Z'(\mathbb{R})\) and \(R(C') \hookrightarrow R(C)\) is a localisation, we have the following facts:

- \(f_{P,Q} \in R(C)\)
- \(\Omega_{R(C')/R} \simeq \Omega_{R(C)/R} \otimes R(C)\) and hence is free
- the components of \(Z(\mathbb{R})\) are contained in the components of \(Z'(\mathbb{R})\)
- if \(z \in Z(\mathbb{R}) \subseteq Z'(\mathbb{R})\), then the corresponding maximal ideal \(m_z \subset R(C')\) satisfies \(R(C')/m_z \simeq R(C)/m_z R(C)\)

If \([P, Q] \subset Z(\mathbb{R})\), then it is called a closed interval of \(Z(\mathbb{R})\). In that case, \(f_{P,Q}\) is positive at all points of \(Z(\mathbb{R})\) outside \([P, Q]\), in particular on all the points outside the component containing \(P\) and \(Q\).

We now define elementary paths.

**Definition 2.3.5.** Let \(X = \text{Spec}(A)\) be an affine variety over \(R\). An elementary path in \(X(\mathbb{R})\) is a totally ordered subset \(\gamma\) of \(X(\mathbb{R})\) which either consists only of one point ("degenerate" elementary path) or has the following two properties:

- The Zariski closure of \(\gamma\) in \(\text{Spec}(A)\) is an irreducible curve \(\text{Spec}(B) \subset \text{Spec}(A)\).
- If \(\Pi : Z = \text{Spec}(C) \to \text{Spec}(B)\) denotes the normalisation of \(\text{Spec}(B)\), then after a choice of a suitable orientation on \(Z(\mathbb{R})\), there exists a bijective and order preserving map from a closed interval \([P, Q] \subset Z(\mathbb{R})\) onto \(\gamma\).

**Remark 2.3.6.** Elementary paths are essentially bijective images of closed intervals in smooth curves onto \(X(\mathbb{R})\). We call \(\Pi(P)\) the starting point or initial point of \(\gamma\) and \(\Pi(Q)\) the endpoint of \(\gamma\).

Note that every elementary path is a bijective image of \([0, 1] \subseteq R\) ([13, Theorem 10.1]). In particular it implies that intervals as defined above for arbitrary smooth curves are also bijective images of \([0, 1]\).

We now quote a theorem that will make it clear why the notion of elementary paths is of importance to us.
Theorem 2.3.7. [13, Theorem 10.2] Any semialgebraic path can be broken into finitely many non-degenerate elementary paths $\gamma_i$, $1 \leq i \leq r$ such that $\gamma_i \cap \gamma_{i+1} = \{S_i\}$ and $S_i$ is the initial point of $\gamma_{i+1}$ and the endpoint of $\gamma_i$.

This result essentially says that the notion of semialgebraic connectedness is an algebraic one, because it means that two points are in the same connected component if and only if one can get a finite sequence of curves such that the first one passes through one of the points and the last one passes through the other, and we can reach one point from the other by traversing through the pieces of curves. This “algebraises” the notion of semialgebraic connectedness.

2.4 Some algebraic results

We prove some results which we will use later. To start with, we prove a lemma which allows us to analyse the natural map $A/J \rightarrow A/\sqrt{J}$ when $ht(J) = \dim(A)$.

**Lemma 2.4.1.** Let $k$ be a field with characteristic $\neq 2$ and let $B$ be a $k$-algebra. Let $I$ be a nilpotent ideal of $B$. Let $g \in B^*$ be such that $g$ has a square root modulo $I$. Then, $\exists g_1 \in B$ such that $g_1^2 = g$. In particular, if $g \equiv 1 \mod I$, then $g_1$ can be so chosen that $g_1 \equiv 1 \mod I$.

**Proof.** Since $I$ is nilpotent, $B$ is complete w.r.t. the $I$-adic topology (which is actually the discrete topology). We attach a variable $Y$ to $B$. Let $f(Y) = Y^2 - g$. Let “bar” denote going modulo $I$. Let $\bar{g} = \bar{h}^2, h \in B$. Since $g$ is a unit, so is $h$. Then,

$$f(Y) = Y^2 - \bar{g} = Y^2 - \bar{h}^2 = (Y - \bar{h})(Y + \bar{h})$$

and since characteristic of $k$ is not equal to 2, $Y - \bar{h}$ and $Y + \bar{h}$ are co-maximal in $(B/I)[Y]$ and hence, applying Hensel's lemma, $Y^2 - g$ has a solution in $B$, say $g_1$ such that $g_1 \equiv \bar{h} \mod I$. If $g \equiv 1 \mod I$, then clearly $g_1 \equiv \pm 1 \mod I$ and hence, we can choose $g_1$ so that $g_1 \equiv 1 \mod I$. \hfill $\square$

The next lemma allows us to analyse conductor diagrams.

**Lemma 2.4.2.** Suppose $f : B \rightarrow B'$ is an injection of rings and let $\mathfrak{c}_{B'/B}$ be the conductor ideal of $B'$ w.r.t. $B$. Let $I$ be an ideal in $B$ such that $I + \mathfrak{c}_{B'/B} = B$. If $\exists f \in B'$ such that $f \equiv 1 \mod \mathfrak{c}_{B'/B}$ and $IB' = fB'$, then $f \in B$ and $I = fB$. 


2.4. SOME ALGEBRAIC RESULTS

Proof. Since \( f \equiv 1 \mod \mathfrak{c}\), \( f - 1 \in \mathfrak{c}\). Let \( f - 1 = x \in \mathfrak{c}\) \( \subseteq B \). Then, \( f = x + 1 \in B \). Further, let \( y \in I \). Then, \( y = fg \), where \( g \in B' \). Then, \( y = (1 + x)g = g + xg \). Now, \( x \in \mathfrak{c}\) \( \Rightarrow xg \in \mathfrak{c}\) \( \subseteq B \). Since \( y \in I \) \( \subseteq B \), we get that \( g \in B \). Hence, \( y = fg \) implies that \( y \in fB \). Hence, \( I = fB \). \( \square \)

We prove another lemma which will be used later to make computations in the Pic group.

**Lemma 2.4.3.** Let \( \mathbb{R} \) be a real closed field. Let \( Z = \text{Spec}(C) \) be a smooth affine curve over \( \mathbb{R} \). If \( \mathfrak{m} \) is a non-real maximal ideal of \( C \), then it always satisfies an equation of the form \( \mathfrak{m} \prod m_i^2(f) = \prod m_i^2(g) \) where \( f \) and \( g \) are sums of squares. In particular, this implies that \( (\mathfrak{m}) \in \text{Pic}(C) \).

Proof. Let \( \overline{\mathbb{R}} \) be the algebraic closure of \( \mathbb{R} \). There is a natural norm map (which is only a multiplicative homomorphism on the units) from \( \mathbb{R} \to \overline{\mathbb{R}} \) given by \( a + bi \mapsto a^2 + b^2 \) which extends to a natural map \( \text{Norm} : C \otimes_\mathbb{R} \overline{\mathbb{R}} \to C \) given by \( f \otimes (a + bi) \mapsto (a^2 + b^2)f^2 \). Let \( \mathfrak{m} \) be a non-real maximal ideal of \( C \). Then there exists a maximal ideal \( \mathfrak{n} \) of \( C \otimes_\mathbb{R} \overline{\mathbb{R}} \) such that \( \mathfrak{n} \cap C = \mathfrak{m} \). It is well-known that \( \text{Pic}(C \otimes_\mathbb{R} \overline{\mathbb{R}}) \) is divisible. Hence,

\[
(n) = 2 \sum_{i=1}^{k_1} s_i(n_i) + 2 \sum_{i=k_1+1}^{k_2} s_i(n_i) - 2 \sum_{i=k_2+1}^{k_3} s_i(n_i) - 2 \sum_{i=k_3+1}^{k_4} s_i(n_i)
\]

where \( s_i > 0 \) and \( n_i \cap C = \mathfrak{m}_i \) where

\[
\mathfrak{m}_i \text{ is a } \begin{cases} \text{real maximal ideal of } C & 1 \leq i \leq k_1, k_2 + 1 \leq i \leq k_3 \\ \text{non-real maximal ideal of } C & k_1 + 1 \leq i \leq k_2 \text{ and } k_3 + 1 \leq i \leq k_4 \end{cases}
\]

Then, this means that there exist \( h_1, h_2 \in C \otimes_\mathbb{R} \overline{\mathbb{R}} \) so that

\[
(h_1)n \prod_{i=k_2+1}^{k_4} n_i^{2s_i} = (h_2) \prod_{i=1}^{k_2} n_i^{2s_i}.
\]

Hence, applying the norm map, we get

\[
\text{Norm}(h_1)\text{Norm}(n) \prod_{i=k_2+1}^{k_4} \text{Norm}(n_i^{2s_i}) = \text{Norm}(h_2) \prod_{i=1}^{k_2} \text{Norm}(n_i^{2s_i})
\]
which gives us

$$(\text{Norm}(h_1))m \prod_{i=k_2+1}^{k_3} m_i^{4s_i} \prod_{i=k_3+1}^{k_4} m_i^{2s_i} = (\text{Norm}(h_2)) \prod_{i=1}^{k_1} m_i^{4s_i} \prod_{i=k_1+1}^{k_2} m_i^{2s_i}.$$ 

Note that $\text{Norm}(h_1)$ and $\text{Norm}(h_2)$ are both sums of squares. Hence, we get the desired result. 

Finally, we state two theorems which loom large over this entire subject. The following theorem of Eisenbud-Evans \cite{15} can be deduced from \cite[p. 1420]{21}.

**Theorem 2.4.4.** Let $A$ be a ring and $P$ be a projective $A$-module of rank $d$. Let $(\alpha, a) \in P^* \oplus A$. Then there exists an element $\beta \in P^*$ such that $\text{ht}(I_a) \geq d$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq d$ then $\text{ht}(I) \geq d$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq d$ and $I$ is a proper ideal of $A$, then $\text{ht}(I) = d$.

When $A$ is a geometrically reduced affine ring we have the following version of Bertini’s theorem (due to Swan) which is a refinement of (2.4.4). This version can be deduced from \cite[Theorems 1.3 and 1.4]{22}; see also \cite{18}.

**Theorem 2.4.5.** Let $A$ be a geometrically reduced affine ring over an infinite field and $P$ be a projective $A$-module of rank $r$. Let $(\alpha, a) \in P^* \oplus A$. Then there exists an element $\beta \in P^*$ such that if $I = (\alpha + a\beta)(P)$ then

1. Either $I_a = A_a$ or $I_a$ is an ideal of height $r$ such that $(A/I)_a$ is a geometrically reduced ring.

2. If $r < \dim A$ and $A_a$ is geometrically integrable, then $(A/I)_a$ is also geometrically integrable.

3. If $A_a$ is smooth, then $(A/I)_a$ is also smooth.
Chapter 3

Elementary Paths in $X(R)$

Let $X = \text{Spec}(A)$ be a smooth affine variety over $R$ of dimension $n \geq 2$. Assume that the set $X(R)$ of real points is not empty, hence infinite. In this chapter, we analyse elementary paths in $X(R)$.

Let $\gamma$ be a non-degenerate elementary path in $X(R)$ as defined in (2.3.5). Then the Zariski closure $\overline{\gamma}$ in $X$ is an irreducible curve. Let $p$ be the prime ideal of $A$ defining this curve and $B = A/p$. Let $C$ be the normalisation of $B$. Then $Z = \text{Spec}(C)$ is a smooth curve. Let $\Sigma_B = \{1 + \sum f_i^2 | f_i \in B\}$. Then $B' = \Sigma_B^{-1}C$ is the normalisation of $R(B)$ and $B' \hookrightarrow R(C)$. $B'$ contains all the real maximal ideals of $C$ and only finitely many non-real maximal ideals (which contract to the singularities of $R(B)$). In particular, that means $R(C) = R(B')$.

Using this, we get:

**Lemma 3.1.6.** Let $m$ be a maximal ideal of $B'$. If $m$ is non-real, then $m$ is principal. If $m$ is real, then $m^2$ is a principal ideal, generated by a sum of squares.

**Proof.** Let $c = c_{B'/R(B)}$ be the conductor of $B'$ over $R(B)$. Now consider the Mayer-Vietoris sequence corresponding to the conductor,

$$U(\frac{B'}{c}) \to \text{Pic}(R(B)) \to \text{Pic}(B') \oplus \text{Pic}(\frac{R(B)}{c}) \to \text{Pic}(\frac{B'}{c}).$$

Since $c$ has height 1, $\text{Pic}(R(B)/c) = \text{Pic}(B'/c) = 0$. Hence, we get $\text{Pic}(R(B)) \to \text{Pic}(B')$. Since $R(B)$ contains only real maximal ideals, by (2.3.1) $\text{Pic}(R(B))$ is
2-torsion. Hence, Pic(B') is also 2-torsion. Putting this together with (2.4.3), we get that every non-real maximal ideal of B' is principal.

Let m be a real maximal ideal of B'. Since B' is a localisation of a smooth, affine curve over R, (2.3.1) gives us \( m^2R(B') = (\sum c_i^2) \) where \( m = (c_1, c_2, \ldots, c_n) \). B' has only finitely many non-real maximal ideals, say \( \mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_k \). Since they are all principal, let \( \prod_{i=1}^k \mathfrak{d}_i = (x) \). Further, choose an element \( y \in m^2 \setminus (\cup_{i=1}^k \mathfrak{d}_i) \). Then consider the element \( z = x^2(\sum c_i^2) + y^2 \). This element clearly does not belong to any real maximal ideal \( m' \) other than \( m \) since

\[
z \in m' \Rightarrow xc_i \in m' \Rightarrow c_i \in m' \Rightarrow m \subseteq m'.
\]

Also the choice of \( y \) means that \( z \notin \mathfrak{d}_i \). Locally, \( z \) generates \( m^2 \) hence we have \( m^2 = (x^2(\sum c_i^2) + y^2) \) which proves the lemma.

This in turn gives us:

**Lemma 3.1.7.** Every non-real maximal ideal \( \mathfrak{d} \) of B' is generated by a function which is positive at all real points, i.e. \( \mathfrak{d} = (h) \) and \( h = h_2/h_1 \) where \( h_1, h_2 \in B' \) and both are sums of squares in B'.

**Proof.** Since \( \mathfrak{d} \) is principal, let \( \mathfrak{d} = (x) \). Then by (2.4.3), we know that there exist \( f \) and \( g \) which are sums of squares such that \( (\mathfrak{d} \cap C) \prod m_j^2(f) = \prod m_j^2(g) \). Hence, we get \( \mathfrak{d} \prod (m_jB')^2(f) = \prod (m_jB')^2(g) \). Hence, by (3.1.6), we get that \( (x)(h_1) = (h_2) \) where \( h_1 \) and \( h_2 \) are sums of squares in B'. This means there exists a unit \( u \in B' \) such that \( uxh_1 = h_2 \). Putting \( h = ux \), we obtain the result.

**Remark 3.1.8.** Since non-real points of B' are principal, \( \text{Pic}(B') \xrightarrow{\sim} \text{Pic}(R(B')) \).

From (2.3.2), there exists a generator \( \chi \) of \( \Omega_{R(C)/R} \) which we fix through the rest of the argument. We summarise the information in the commutative diagrams below:

\[
\begin{array}{c}
B' \xleftarrow{\xi} C' \\
\downarrow \\
R(B) \xrightarrow{\Sigma_B^{-1}C} R(C)
\end{array}
\]

(∗∗∗1)
which along with the definition of elementary path gives us:

\[ \gamma \xleftarrow{\sim} \text{(Spec}(B))(R) \xrightarrow{\sim} \text{Max}(R(B)) \quad (**2) \]

Consider \( \xi^*(P) \) and \( \xi^*(Q) \) which are elements of \( \text{Max}(R(B)) \hookrightarrow X(R) \) and let \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) be the corresponding maximal ideals of \( R(A) \). Then \( \mathcal{M}_0 \) is the starting point of \( \gamma \) and \( \mathcal{M}_1 \) is the endpoint. Let \( m_0, m_1 \) be the maximal ideals of \( R(B) \) corresponding to \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \). Let \( m_P \) and \( m_Q \) be the maximal ideals of \( B' \) corresponding to \( P \) and \( Q \) respectively. Recall that (2.3.3) gave us a function \( f_{P,Q} \in R(C) \) with special properties.

**Lemma 3.1.9.** We can choose \( g_{P,Q} \) in \( B' \) such that:

1. \( (g_{P,Q}) = m_P \cap m_Q \) in \( B' \).
2. if \( dg_{P,Q} = t \chi \), then it has opposite orientations at both points, i.e. \( \text{sign}(t(P)) = -1, \text{sign}(t(Q)) = 1 \).
3. \( g_{P,Q} \) is positive at all points outside the closed interval \([P,Q] \).

**Proof.** Since \( R(C) \xrightarrow{\sim} R(B') \), using (2.3.3) there exists a function \( f_{P,Q} \in R(B') \) satisfying \( (f_{P,Q}) = m_P R(B') \cap m_Q R(B') \) and \( \text{sign}(g(P)) = -1, \text{sign}(g(Q)) = 1 \) where \( df_{P,Q} = g \chi \). Further, \( f_{P,Q} \) was positive outside the closed interval \([P,Q] \).

Since \( R(B') \) is a localisation of \( B' \), we have \( f_{P,Q} = f/u \) where \( u = 1 + \sum a_i^2 \) and \( f, u, a_i \in B' \). Note that since \( u \) is a sum of squares, \( \text{sign}(f_{P,Q}(R)) = \text{sign}(f(R)) \) for all points \( R \in (\text{Spec } B')(R) \). Let \( df = g_1 \chi \). Since \( f_{P,Q}(P) = f_{P,Q}(Q) = 0 \), we have \( g_1(P) = u(P)g(P) \) and \( g_1(Q) = u(Q)g(Q) \) and hence, \( \text{sign}(g_1(P)) = -1 \) and \( \text{sign}(g_1(Q)) = 1 \) so they continue to have opposite orientations.

By (3.1.8), \( \exists h \in B' \) such that \( m_P \cap m_Q = (h) \). Denote the non-real maximal ideals of \( B' \) by \( \mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_k \). By (3.1.7), \( \exists f_i, 1 \leq i \leq k \) such that \( \mathfrak{d}_i = (f_i) \) where \( f_i \) is positive at all real points of \( B' \). Then, \( f = u_1 \prod_{i=1}^k f_i^{r_i} h \) where \( u_1 \) is a unit in \( B' \). Let \( g_{P,Q} = u_1 h \). Since \( u_1 \) is a unit, \( (g_{P,Q}) = m_P \cap m_Q \). Moreover we have
\[ f = \prod_{i=1}^{k} f_i^{r_i}, g_{P,Q}. \] Hence, \( \text{sign}(f(R)) = \text{sign}(g_{P,Q}(R)) \forall R \in (\text{Spec}(B'))(R) \) and so, \( g_{P,Q} \) is positive at all points outside the closed interval \([P, Q]\). Also, if \( dg_{P,Q} = t\chi \), then \( \text{sign}(t(P)) = \text{sign}(g_1(P)) = -1 \) and \( \text{sign}(t(Q)) = \text{sign}(g_1(Q)) = 1 \) as argued previously. This completes the proof. \( \Box \)

Now we prove a lemma which says that if \( \gamma \) contains only non-singular points of \((\text{Spec}(B'))(R)\), then \( m_0 \cap m_1 \) is a complete intersection in \( R(B) \). In what follows, for \( T \in (\text{Spec}(B'))(R) \), we denote the corresponding maximal ideal of \( R(B) \) by \( m_T \).

**Lemma 3.1.10.** Suppose \( \gamma \) contains only non-singular points of \((\text{Spec}(B'))(R)\). Then given a finite set \( \{T_1, \ldots, T_r\} \cup \{T_1', \ldots, T_s'\} \) of points in \((\text{Spec}(B'))(R)\) not contained in \( \gamma \), there exists \( g \in R(B) \) such that

1. \( (g) = m_0 \cap m_1 \)
2. \( g - 1 \in J \) where \( J = (\cap_{i=1}^{r} m_{T_i}) \cap (\cap_{j=1}^{s} m_{T_j'}) \cap \epsilon_{B'/R(B)} \)
3. \( \text{sign}(dg/\chi) = -1 \) at \( m_0 \) and \( \text{sign}(dg/\chi) = 1 \) at \( m_1 \).

**Proof.** Note that without loss of generality, we can assume that points \( T_i \) and \( T_j' \) correspond to smooth maximal ideals of \( R(B) \), and hence \( T_i, T_j' \) can be regarded as points of \((\text{Spec}(B'))(R)\) as well (since \( (\xi^*)^{-1}(T_i) \) and \( (\xi^*)^{-1}(T_j') \) are singleton). Consider the function \( g_{P,Q} \in B' \) as in (3.1.9) and any point \( T \in (\text{Spec}(B'))(R) \) such that the corresponding maximal ideal \( m_T \) contains \( JB' \). Then either it is \( T_i \) or \( T_j' \) or it is in the support of the conductor ideal \( \epsilon_{B'/R(B)} \). Since none of these points are contained in \([P, Q]\) (since none of the images under \( \xi^* \) are contained in \( \gamma \)), we have that \( g_{P,Q}(T) > 0 \). Consider \( \bar{g}_{P,Q} \in (B'/J)^* \). Then, applying (2.4.1) (with \( I = \sqrt{J}/J \)), we get \( g_1 \in B' \) such that \( \bar{g}_1^2 = \bar{g}_{P,Q} \in (B'/J)^* \) and \( (g_1) + J = B' \). Hence, there exists \( a \in B', x \in J \) such that \( ag_1 + x = 1 \in B' \). Note that \( (g_1, x) = B' \). Consider \( u = g_1^2 + x^2 \). Let \( m \) be a maximal ideal of \( B' \). If \( J \subseteq m \), then \( x \in m \). Therefore, \( g_1 \not\in m \) and hence \( u \not\in m \). If \( J \not\subseteq m \), then \( \epsilon_{B'/R(B)} \not\subseteq m \) and hence if \( n = m \cap R(B) \), then \( R(B)n = B'_m \). As \( R(B) \) has only real maximal ideals, we get that \( m \) is a real maximal ideal of \( B' \). Hence, \( u \in m \Rightarrow g_1, x \in m \Rightarrow 1 \in m \) which is a contradiction. Therefore, \( u \) is a unit in \( B' \) and since it is a sum of squares, it is positive at all real points. Consider
\( g = u^{-1}g_{P,Q}. \) Therefore, with bar denoting elements in \( B'/J, \)
\[ \bar{g} = u^{-1}\bar{g}_{P,Q} = \bar{u}^{-1}\bar{g}_{P,Q} = \bar{g}_1^{-1}\bar{g}_{P,Q} = \bar{g}_1^{-2} = 1 \in B'/J. \]

Hence, \( g = 1 + y \) for some element \( y \in J \subseteq \mathfrak{c}_{B'/R(B)}. \) Hence, using (2.4.2), we get \( m_0 \cap m_1 = (g) \). Further, we see that at the points \( T_i, T'_j, g(T_i) = g(T'_j) = 1 \) and \( g(T) = 1 \) for each point \( T \in (\text{Spec}(B))(R) \) which is a singular point.

Note that
\[ \frac{\Omega_{R(B)/R}}{m_0\Omega_{R(B)/R}} \sim \frac{\Omega_{B'/R}}{m_P\Omega_{B'/R}} \]
and
\[ \frac{\Omega_{R(B)/R}}{m_1\Omega_{R(B)/R}} \sim \frac{\Omega_{B'/R}}{m_Q\Omega_{B'/R}} \]

Since \( g = u^{-1}g_{P,Q} \) in \( B' \), we have \( dg = u^{-1}dg_{P,Q} + g_{P,Q}d(u^{-1}) \) in \( \Omega_{B'/R}. \) Hence,
\[ (dg/\chi)(P) = u^{-1}(P)(dg_{P,Q}/\chi)(P), \quad (dg/\chi)(Q) = u^{-1}(Q)(dg_{P,Q}/\chi)(Q). \]

Therefore, \( \text{sign}((dg/\chi)(P)) = -1 \) and \( \text{sign}((dg/\chi)(Q)) = 1. \)

Let \( L \) be a projective \( A \)-module of rank 1. We prove the final lemma of this chapter which shows that if \( \gamma \) contains only non-singular points of \( R(B) \), then there exists \( \beta : R(L) \oplus R(A)^{n-1} \to \mathcal{M}_0 \cap \mathcal{M}_1. \)

We recall the set-up once again. Recall that \( B = A/p = \overline{\gamma}^{\text{Zar}} \) where \( \gamma \) is an elementary path in \( X(R) \) with starting point \( \mathcal{M}_0 \) and endpoint \( \mathcal{M}_1. \) Let \( C \) be the normalisation of \( B, B' \) be the normalisation of \( R(B) \) and hence \( R(B') = R(C). \)

\( \chi \) is a generator of \( \Omega_{R(C)/R} = \Omega_{R(B')/R}. \) Let \( \mathfrak{c}_{B'/R(B)} \) be the conductor ideal of \( B' \) over \( R(B). \) Let \( \mathfrak{c} \supseteq pR(A) \) be the ideal of \( R(A) \) such that \( \mathfrak{c}/pR(A) = \mathfrak{c}_{B'/R(B)}. \)

Since \( A \) is smooth, \( pR(A)p = (a_1, a_2, \ldots, a_{n-1}) \) with \( a_i \in R(A). \) Let \( B_1 = R(A)/(a_1, a_2, \ldots, a_{n-1}). \) Note that \( (a_1, a_2, \ldots, a_{n-1}) = pR(A) \cap I \) for some ideal \( I \) of \( R(A) \) not contained in \( pR(A). \)

Hence onward we use the following convention. For a maximal ideal \( \mathcal{M} \) of \( R(A) \) containing \( pR(A), \overline{\mathcal{M}} \) denotes the corresponding maximal ideal of \( B_1 \) and \( m \) will denote the corresponding maximal ideal of \( R(B), \) i.e. \( \overline{\mathcal{M}} = MB_1 \) and \( m = MB \). Since \( pR(A) + I \) is of height \( n \) in \( R(A), \) there are finitely many maximal ideals containing \( pR(A) + I. \) Let those be denoted by \( T_1, \ldots, T_r \) in \( X(R). \)
Lemma 3.1.11. Suppose, with the assumptions and notations as above, there exists \( f \in R(A) \) such that:

1. \( f \in C \cap (pR(A) + I) \).
2. \( \text{Spec}(R(A)/f) \cap \gamma = \emptyset \).

Then, there exists \( a_n \in R(A) \) such that \( (a_1, a_2, \ldots, a_n) = M_0 \cap M_1 \), \( a_n - 1 \in (a_1, a_2, \ldots, a_{n-1}, f) \) and \( \text{sign}((d(a_n)/\chi)(P)) = -1 \) and \( \text{sign}((d(a_n)/\chi)(Q)) = 1 \) where \( a_n \) denotes the image of \( a_n \) under the map \( R(A) \to R(C) \). Moreover, if there exists \( \tau \in L \) such that \( R(L)_f = \tau R(L)_f \), then there exists a surjection \( \beta : R(L) + R(A)^{n-1} \to M_0 \cap M_1 \) such that \( \beta(\tau) = a_1 + h a_n \) and \( \beta(e_i) = a_i \); \( 2 \leq i \leq n \) where \( (e_2, \ldots, e_n) \) denotes a basis of \( R(A)^{n-1} \).

Proof. Since \( \text{Spec}(R(A)/f) \cap \gamma = \emptyset \), we have \( f \notin p \).

Let \( \Upsilon = \{ t \in X(R)|p + f \in M_T \} = V((p + f)R(A)) \),

\[
\{ T'_1, \ldots, T'_s \} = \Upsilon \setminus V(C \cap (pR(A) + I)),
\]

\[
\{ T_1, \ldots, T_r \} = V(pR(A) + I) \setminus V(C) \quad \text{and}
\]

\[
J = (pR(A) + I) \cap C \cap (\cap_{i=1}^s M_{T'_i}) \subset R(A).
\]

Then \( \sqrt{J} = (\cap_{i=1}^s M_{T'_i}) \cap \sqrt{C} \cap (\cap_{j=1}^r M_{T_j}) = \bigcap_{T \in \Upsilon} M_T = \sqrt{pR(A) + (f)} \).

Note that since \( pB_1 \cap IB_1 = 0 \), the above equality implies

\[
\sqrt{IB_1} \cap \sqrt{pB_1} \cap (\cap_{i=1}^s M_{T'_i}) \subseteq \sqrt{(f)B_1}.
\]

Moreover, as \( \gamma \subset V(pR(A)) \) and \( \text{Spec}(R(A)/f) \cap \gamma = \emptyset \), \( \gamma \cap \Upsilon = \emptyset \). By (3.1.10), there exists \( g \in R(B) \) such that

- \( (g) = m_0 \cap m_1 \)
- \( g - 1 \in \sqrt{JR(B)} = \sqrt{J}/pR(A) \)
- as an element of \( R(C)(\supset R(B)) \), \( \text{sign}(dg/\chi) = -1 \) at \( m_P (= m_0 R(C)) \) and \( \text{sign}(dg/\chi) = 1 \) at \( m_Q (= m_1 R(C)) \).

Consider the ring \( B'_1 = R(B) + R(A)/I \). There is a natural surjection \( B'_1 \to R(B) \) and through this, we have \( (m_0 \cap m_1) + R(A)/I = ((g, 1)) \) in \( B'_1 \). The natural map \( B_1 \to R(B) \) factors through as \( B_1 \leftrightarrow B'_1 \to R(B) \). Note that the conductor
\[ c_{B_1} = \overline{pR(A)} + I \quad \text{in} \quad B_1 \text{ maps bijectively to} \quad (pR(A) + I)/pR(A) \oplus (pR(A) + I)/I \quad \text{in} \quad B'_1. \]

Since \( g - 1 \in \sqrt{JR(B)} = \sqrt{J/pR(A)} \), the equation \( Y^2 - g \) has a solution in \( R(B)/\sqrt{JR(B)} \). Hence, by (2.4.1), there exists \( g_1 \in R(B) \) such that \( g_1^2 - g \in JR(B) \). As \( (g) + JR(B) = R(B) \), \( (g_1) + JR(B) = R(B) \). Let \( y \in J \) such that \( (g_1) + (y) = R(B) \). Then, \( v = g_1^2 + y^2 \) is a unit in \( R(B) \). Consider the element \( g_2 = v^{-1}g \). Then, \( (g_2) = m_0 \cap m_1 \) and \( g_2 - 1 \in JR(B) \). Further, since \( v \) is a sum of squares and a unit, we get that

\[
\text{sign}\left(\frac{d_{g_2}}{\chi}(P)\right) = \text{sign}\left(\frac{dg}{\chi}(P)\right) = -1 \quad \text{and}
\]

\[
\text{sign}\left(\frac{d_{g_2}}{\chi}(Q)\right) = \text{sign}\left(\frac{dg}{\chi}(Q)\right) = 1.
\]

Note that since \( pR(A) \subseteq J \subseteq pR(A) + I \) and \( g_2 - 1 \in JR(B) = J/pR(A) \), the element \( (g_2, 1) - (1, 1) \) of \( R(B) \oplus R(A)/I(= B'_1) \) belongs to the conductor ideal \( c_{B'_1/B_1} = (pR(A) + I)/pR(A) \oplus (pR(A) + I)/I \subseteq B'_1 \). Hence, there exists \( b \in B_1 \) such that \( b \mapsto (g_2, 1) \). We note that the hypothesis implies \( \overline{M_0} \cap \overline{M_1} \not\subseteq \overline{I} \). Hence, \( (\overline{M_0} \cap \overline{M_1})B'_1 = (m_0 \cap m_1) \oplus R(A)/I = (g_2, 1)B'_1 \) and therefore, by (2.4.2), \( b = \overline{M_0} \cap \overline{M_1} \). As above, we have

\[
\text{sign}\left(\frac{d_{\tilde{b}}}{\chi}(P)\right) = \text{sign}\left(\frac{d_{g_2}}{\chi}(P)\right) = -1 \quad \text{and}
\]

\[
\text{sign}\left(\frac{d_{\tilde{b}}}{\chi}(Q)\right) = \text{sign}\left(\frac{d_{g_2}}{\chi}(Q)\right) = 1
\]

where \( \tilde{b} \) is the image of \( b \) in \( R(C) \).

Since \( \tilde{f} \notin \overline{M_0} \cap \overline{M_1} \), \( (\tilde{f})B_1 + (b)B_1 = B_1 \). Further,

\[
b - 1 \mapsto (g_2 - 1, 0) \in \sqrt{JR(B)} \oplus R(A)/I
\]

\[
\Rightarrow b - 1 \in \sqrt{IB_1} \cap cB_1 \cap (\cap_{j=1}^{s} \overline{M_{T_j}}) \subseteq \sqrt{IB_1}.
\]

Hence, the equation \( Y^2 - b \) has a solution in \( B_1/\sqrt{IB_1} \) and therefore, by (2.4.1), we have \( a \in B_1 \) such that \( a^2 - b \in fB_1 \). As \( b \equiv 1 \mod \sqrt{IB_1} \), \( (a, \tilde{f}) = B_1 \) where \( \tilde{f} \) is the image of \( f \) in \( B_1 \). Therefore, \( u = a^2 + \tilde{f}^2 \) is a unit in \( B_1 \). Consider \( c \in R(A) \)
such that \( \tilde{c} = u^{-1}h \). Then, \( \tilde{c} - 1 \in \langle \tilde{f} \rangle \). Hence, \( c - 1 = \alpha f + \sum_{j=1}^{n-1} \alpha_j a_j \). Let \( a_n = c - \sum_{j=1}^{n-1} \alpha_j a_j \). Therefore,

\[
a_n - 1 = \alpha f \Rightarrow (a_n, f) = R(A).
\]

Further, \( (\bar{a}_n) = (\bar{c}) = \bar{M}_0 \cap \bar{M}_1 \) where “bar” denotes the image in \( B_1 \) and hence \( (a_1, a_2, \ldots, a_{n-1}, a_n) = M_0 \cap M_1 \) in \( R(A) \). Also,

\[
\text{sign}(\frac{d(\tilde{a}_n)}{\chi})(P) = \text{sign}(\frac{d(\tilde{c})}{\chi})(P) = \text{sign}(\frac{d(\tilde{b})}{\chi})(P) = -1 \quad \text{and}
\]

\[
\text{sign}(\frac{d(\tilde{a}_n)}{\chi})(Q) = \text{sign}(\frac{d(\tilde{c})}{\chi})(Q) = \text{sign}(\frac{d(\tilde{b})}{\chi})(Q) = 1
\]

where “tilde” are the images in \( R(C) \).

Now assume that there exists \( \tau \in L \) such that \( R(L)\tau = \tau R(A)\). Since \( R(A)f + R(A)a_n = R(A) \), we have \( R(L)/a_n R(L) = R(L)f/a_n R(L)f \). Therefore as \( R(L)\tau = \tau R(A)\), \( R(L)/a_n R(L) \) is a free \( R(A)/(a_n) \)-module of rank 1 with \( \bar{\tau} \) as a generator. Therefore, it is easy to see that there exists an \( R(A) \)-linear map \( \alpha : R(L) \rightarrow M_0 \cap M_1 \) such that \( \alpha(\tau) = a_1 + a_n h \) with \( h \in R(A) \).

Let

\[
\beta : R(L) \oplus R(A)^{n-1} \rightarrow M_0 \cap M_1
\]

be an \( R(A) \)-linear map map defined as \( \beta(l) = \alpha(l) \) for \( l \in R(L) \) and \( \beta(e_i) = a_i \); \( 2 \leq i \leq n \) where \( (e_2, \ldots, e_n) \) is a basis of \( R(A)^{n-1} \). Then, \( \beta(\tau - he_n) = a_1 + ha_n - ha_n = a_1 \). Hence, as \( M_0 \cap M_1 = (a_1, a_2, \ldots, a_n) \), \( \beta \) is a surjection. Thus, we obtain the result. \( \square \)
Chapter 4

Elementary Paths in $Z(\mathbf{R})$

Let $X = \text{Spec}(A)$ be a smooth affine variety over $\mathbf{R}$ of dimension $n \geq 2$. Assume further that the set $X(\mathbf{R})$ of real points is not empty, hence infinite. Let $L$ be a projective $A$-module of rank 1. We denote $K_A = \wedge^n(\Omega_A)_{\mathbf{R}}$ by $K$.

Let $\mathcal{E} = L \otimes_A K$. Let $D = \bigoplus_{-\infty < i < \infty} \mathcal{E}^i$. Let $Z = \text{Spec}(D)$. Then there is a natural map $A \hookrightarrow D$ which gives rise to a natural surjection $Z \twoheadrightarrow X$ which induces a natural map $Z(\mathbf{R}) \twoheadrightarrow X(\mathbf{R})$ which we denote by $\Pi$. Looked at in the Euclidean topology, this gives a semialgebraic $\mathbf{R}^n$-bundle over $X(\mathbf{R})$ (actually it is better than just semialgebraic but we are content with this structure).

In what follows, we identify points of $Z(\mathbf{R})$ (respectively $X(\mathbf{R})$) with the corresponding real maximal ideals of $D$ (respectively $A$). Recall that $\mathbf{R}(A)$ denotes the ring obtained from $A$ by inverting all elements of the type $1 + \sum_{i=1}^n f_i^2$; $f_i \in A$ and $\mathbf{R}(L) = L \otimes_A \mathbf{R}(A)$. Let

$$Y = \{ (\mathcal{M}, \omega_{\mathcal{M}}) | \mathcal{M} \in X(\mathbf{R}), \omega_{\mathcal{M}} : \frac{L}{ML} \to \wedge^n(\frac{\mathcal{M}}{\mathcal{M}^2}) \}.$$

Recall that the Euler class group $E(\mathbf{R}(A), \mathbf{R}(L))$ is a quotient of the free abelian group with generating set $Y$. We associate with $Y$ the topological space $Z(\mathbf{R})$ as follows:

Let $\mathcal{M}$ be a real maximal ideal. Let

$$Y_{\mathcal{M}} = \{ (\mathcal{M}, \omega_{\mathcal{M}}) | \omega_{\mathcal{M}} : \frac{L}{ML} \to \wedge^n(\frac{\mathcal{M}}{\mathcal{M}^2}) \}.$$
The differential map $d : A \to \Omega_{A/R}$ induces $\wedge^n(d_M) : \wedge^n(M/M^2) \sim K/MK$. Composing $\omega_M$ with $\wedge^n(d_M)$, we get an isomorphism $\phi_{\omega_M} : L/ML \sim K/MK$. Note that

$$\frac{R(E)}{MR(E)} = \frac{R(L)}{MR(L)} \otimes \frac{R(K)}{MR(K)}$$

and hence there is a natural isomorphism $R(L)^2/\text{MR}(L)^2 \to R(E)/\text{MR}(E)$ given by

$$\Gamma_{\omega_M}(l \otimes l') = l \otimes \phi_{\omega_M}(l') ; \quad l, l' \in \frac{L}{ML}.$$

Thus, we get:

$$\text{Hom} \left( \frac{R(L)}{\text{MR}(L)}, \wedge^n(M/M^2) \right) \xrightarrow{\sim} \text{Hom} \left( \frac{R(L)}{\text{MR}(L)}, \frac{R(K)}{\text{MR}(K)} \right) \xrightarrow{\Gamma} \text{Hom} \left( \frac{R(L)^2}{\text{MR}(L)^2}, \frac{R(E)}{\text{MR}(E)} \right),$$

$\omega_M \mapsto \wedge^n(d_M) \circ \omega_M \mapsto \text{id}_{R(L)} \otimes \wedge^n(d_M) \circ \omega_M$.

By (2.3.1), $R(L)^2$ is free. Let $\kappa \in L^2$ be a generator of $R(L)^2$. Let $\pi$ denote the image of $\kappa$ in $R(L)^2/\text{MR}(L)^2$. Then $\Gamma_{\omega_M}(\pi)$ is a non-zero element of $R(E)/\text{MR}(E)(= E/ME)$ and hence $\Gamma_{\omega_M}(\pi) = \bar{e}$, where $e \in E \setminus ME$. Then, this gives a map $\Theta_M : Y_M \to \Pi^{-1}(M)$, sending $(M, \omega_M) \mapsto (\bar{M}, \bar{e} - 1)$ where $e$ is defined as above. Now, every element of $\Pi^{-1}(M)$ is of the form $(\bar{M}, \bar{e} - 1)$ where $e \in E \setminus ME$. Hence, given $e$, we get an isomorphism sending $\bar{e}$ to $\bar{e}$ and working backwards in the above diagram, we get a local orientation of $M$. Hence, $\Theta_M$ is a bijection.

Note that there is a natural action of $R^*$ on $Y_M$ and $\Pi^{-1}(M)$ and the above diagram shows that the map $\Theta_M$ is compatible with the action. Putting together $\Theta_M$ for all $M \in X(R)$, we have $\Theta : Y \sim Z(R)$. Therefore, we get a set-theoretic map $Z(R) \to E(R(A), R(L))$.

In this chapter, using elementary paths in $Z(R)$ we show that : \textit{image of a component of $Z(R)$ under the map $Z(R) \to E(R(A), R(L))$ is singleton.}

To show this we need to prove some auxiliary results. We first set up notations required for these results.

Suppose $f \in A$ is such that $L_f \simeq A_f \simeq K_f$. Let us fix generators $\tau$ and $\rho$ of $L_f$ and $K_f$ respectively. Let $Z_f = \text{Spec}(D_f)$ and $X_f = \text{Spec}(A_f)$. Then $E_f$ is generated by $\tau \otimes \rho$ and therefore

$$D_f = D \otimes_A A_f = A_f[T, T^{-1}]; \quad T = (\tau \otimes \rho).$$
With respect to the pair \((\tau, \rho)\), we assign to every \(P \in Z_f(\mathbb{R})\), an element of the group \(\{1, -1\}\) as follows:

**Definition 4.1.12.** Let \(\Theta(M, \omega_M) = P\) correspond to \((M, e - 1)\) where \(e \in \mathcal{E}, \bar{e} \neq 0 \in \mathcal{E}/M\mathcal{E}.\) Then if \(f \notin M\), we have

\[
\bar{e} = \lambda \tau \otimes p : \lambda \in \mathbb{R}^*.
\]

Define

\[
\text{sgn}(\tau, \rho)(P) = \text{sgn}(\tau, \rho)(M, \omega_M) = \text{sign}(\lambda).
\]

**Remark 4.1.13.** Since \(D_f = A_f[T, T^{-1}], T = (\tau \otimes \rho)\), we can consider \(T\) as a function on \(Z_f(\mathbb{R})\). Let \((M, \omega_M) \in Y\) such that \(P = \Theta((M, \omega_M)) \in Z_f(\mathbb{R}).\) Then the value of \(T\) at \(P\) is given by \(\lambda^{-1} \in \mathbb{R}^*\) where \(\tilde{T} = \lambda^{-1} \bar{e} = \lambda^{-1} \Gamma_{\omega_M}(\tau) \in \mathcal{E}/(M\mathcal{E}).\) This implies that

\[
\text{sgn}(\tau, \rho)(M, \omega_M) = \text{sign}(T(P)).
\]

This further implies that if \(\omega_M, \tilde{\omega}_M\) are two local \(L\)-orientations of \(M\), then

\[
\text{sgn}(\tau, \rho)(M, \tilde{\omega}_M) = \text{sign}(\alpha) \text{sgn}(\tau, \rho)(M, \omega_M) \quad \text{where} \quad \tilde{\omega}_M = \alpha \omega_M.
\]

Since \(\mathcal{E}_f\) is free, \(Z_f(\mathbb{R}) \simeq X_f(\mathbb{R}) \times \mathbb{R}^+ \bigsqcup X_f(\mathbb{R}) \times \mathbb{R}^-\) where the maximal ideal \((M, T - 1)\) corresponds to \((M, 1) \in X_f(\mathbb{R}) \times \mathbb{R}^+\). Note that \(P \in X_f(\mathbb{R}) \times \mathbb{R}^+\) if and only if \(\text{sgn}(\tau, \rho)(P) = 1\).

Recall that we have a (set-theoretic) map \(Z(\mathbb{R}) \to E(\mathbb{R}(A), \mathbb{R}(L)).\) For \(P \in Z(\mathbb{R})\), we denote its image in \(E(\mathbb{R}(A), \mathbb{R}(L))\) by \((P)\). Let \(\Psi\) be an elementary path in \(Z(\mathbb{R})\) and let \(P\) be the starting point and \(Q\) be the endpoint of \(\Psi\). Our first step is to show that under the map \(Z(\mathbb{R}) \to E(\mathbb{R}(A), \mathbb{R}(L)), (P) = (Q)\) in \(E(\mathbb{R}(A), \mathbb{R}(L)).\)

**Lemma 4.1.14.** Let \(\Psi\) be an elementary path in \(Z(\mathbb{R})\) such that under the canonical map \(\Pi : Z(\mathbb{R}) \to X(\mathbb{R}), \Pi(\Psi)\) is a singleton. Let \(P = \Theta((M_0, \omega_{M_0}))\) be the starting point and \(Q = \Theta((M_1, \omega_{M_1}))\) be the end point of \(\Psi\). Then, \((M_0, \omega_{M_0}) = (M_1, \omega_{M_1})\) in \(E(\mathbb{R}(A), \mathbb{R}(L)).\)
Proof. Since \( \Pi(\Psi) \) is singleton, \( M_0 = \mathcal{M}_1 \) and \( \Psi \subset \Pi^{-1}(M_0) \). Choose \( f \notin M_0 \) such that \( L_f \simeq A_f \simeq K_f \). Then, \( M_0 = \mathcal{M}_1 \) and hence, \( \omega_{\mathcal{M}_0} = \lambda \omega_{\mathcal{M}_1} \). Then, choosing generators \( \tau \) and \( \rho \) for \( L_f \) and \( K_f \) respectively, we can express \( \Pi^{-1}(M_0) \) as a union of its components \( \Pi^{-1}(M_0) = M_0 \times \mathbf{R}^+ \biguplus M_0 \times \mathbf{R}^- \). Since \( \Psi \) is semialgebraically connected, \( \Psi \) lies in one of them. Hence, \( \text{sgn}((\tau,\rho)) \) \( M_0, \omega_{\mathcal{M}_0} = \text{sgn}(\tau,\rho)(M_1, \omega_{\mathcal{M}_1}) \) and hence, \( \text{sgn}(\lambda) > 0 \). But then by (2.1.5), this implies \( (M_0, \omega_{\mathcal{M}_0}) = (M_1, \omega_{\mathcal{M}_1}) \) in \( E(\mathbf{R}(A), \mathbf{R}(L)) \), i.e. \( (P) = (\mathcal{Q}) \).

Now we assume that \( \gamma = \Pi(\Psi) \) is not singleton and moreover that \( \Pi|_{\Psi} : \Psi \to \gamma \) is bijective. Therefore, by [12, Theorem 3.1], \( \gamma \) is a nondegenerate elementary path in \( X(\mathbf{R}) \). We set up some notations and prove some results to deal with this case.

Let \( \mathcal{Z}_{\text{zar}} = \text{Spec}(D/\mathfrak{q}) \) and \( \tilde{\gamma}_{\text{zar}} = \text{Spec}(A/\mathfrak{p}) \). Then, as \( \gamma \) is non-degenerate, \( \mathfrak{p} = \mathfrak{q} \cap A \).

In this context, we recall the notations used in the previous chapter : \( B = A/\mathfrak{p}A \). Let \( C \) be the normalisation of \( B \) and \( \xi : B \leftarrow C \). Let \( B' \) be the normalisation of \( \mathbf{R}(B) \) and hence \( \mathbf{R}(B') = \mathbf{R}(C) \). \( \chi \) is a generator of \( \Omega_{\mathbf{R}(C)/\mathbf{R}} = \Omega_{\mathbf{R}(B')/\mathbf{R}} \). Since \( \gamma \) is an elementary path, we have an order preserving bijection \( \xi^*: [P, Q] \rightleftharpoons \gamma \) where \( [P, Q] \subset (\text{Spec}(C))(\mathbf{R}) \). Then, \( \xi^*(P) \) and \( \xi^*(Q) \) are the start and end points of \( \gamma \) respectively. Let \( M_0 \) and \( M_1 \) be the maximal ideals of \( \mathbf{R}(A) \) corresponding to \( \xi^*(P) \) and \( \xi^*(Q) \) respectively.

Let \( \mathfrak{c}_{B'/\mathbf{R}(B)} \) be the conductor ideal of \( B' \) over \( \mathbf{R}(B) \). Let \( \mathfrak{c} \) be the ideal of \( \mathbf{R}(A) \) containing \( \mathfrak{p}\mathbf{R}(A) \) such that \( \mathfrak{c}/\mathfrak{p}\mathbf{R}(A) = \mathfrak{c}_{B'/\mathbf{R}(B)} \). Since \( \mathbf{R}(A) \) is regular, \( \mathfrak{p}\mathbf{R}(A) = (a_1, a_2, \ldots, a_{n-1}), a_i \in \mathfrak{p}\mathbf{R}(A) \). Then, \( (a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p}\mathbf{R}(A) \cap I \) for some ideal \( I \subset \mathbf{R}(A) \) not contained in \( \mathfrak{p}\mathbf{R}(A) \). Let \( B_1 = \mathbf{R}(A)/(a_1, a_2, \ldots, a_{n-1}) \).

In what follows, we shall use the following convention. For a maximal ideal \( \mathcal{M} \) of \( \mathbf{R}(A) \) containing \( \mathfrak{p}\mathbf{R}(A) \), \( \overline{\mathcal{M}} \) will denote the corresponding maximal ideal of \( B_1 \) and \( \mathfrak{m} \) will denote the corresponding maximal ideal of \( \mathbf{R}(B) \), i.e. \( \overline{\mathcal{M}} = \mathcal{M}B_1 \) and \( \mathfrak{m} = \mathcal{M}\mathbf{R}(B) \).

Since \( I \notin \mathfrak{p}\mathbf{R}(A) \), \( \mathfrak{p}\mathbf{R}(A) + I \) is of height \( n \) in \( \mathbf{R}(A) \), and hence \( (\mathfrak{p}\mathbf{R}(A) + I) \cap \mathfrak{c} \) is an ideal of height \( n \).

We now prove the following lemma :
Lemma 4.1.15. Let $\Psi \subseteq Z(R)$ be a nondegenerate elementary path and $\gamma = \Pi(\Psi)$. Suppose $\Pi : \Psi \rightarrow \gamma$ is a bijection. Let $(M_0, \omega_{M_0})$ and $(M_1, \omega_{M_1})$ be such that $\Theta((M_0, \omega_{M_0})) = \mathcal{P}$ and $\Theta((M_1, \omega_{M_1})) = Q$ are initial and end points of $\Psi$ respectively. Further, assume that, with notation as above, there exists $f \in R(A)$ such that:

1. $f \in \mathcal{C} \cap (pR(A) + I)$.
2. $L_f \simeq A_f \simeq K_f$.
3. $\text{Spec}(R(A)/f) \cap \gamma = \emptyset$.

Then, $(M_0, \omega_{M_0}) = (M_1, \omega_{M_1})$ in $E(R(A), R(L))$, i.e. $\mathcal{P} = (Q)$.

Proof. Let

$$\mathcal{Y} = \{ T \in X(R)|p + f \in M_T \} = V((p + f)R(A)),$$

$$\{ T_1', \ldots, T_s' \} = \mathcal{Y} \setminus V(\mathcal{C} \cap (pR(A) + I)).$$

Note that

$$\sqrt{TB_1} \bigcap \mathcal{C} \bigcap (\cap_{j=1}^{s} M_{T_j'}) \subseteq \sqrt{(f)B_1}.$$

Also, $\gamma \subseteq X_f(R)$. Further, $R(B)_f \simeq B'_f$ since the conductor $(c_{B'/R(B)})_f$ equals the full ring $B'_f$. So, $R(B)_f$ is regular and hence, we have the short exact sequence,

$$0 \rightarrow \frac{(a_1, a_2, \ldots, a_n)R(A)_f}{p(a_1, a_2, \ldots, a_n)R(A)_f} \rightarrow \frac{\Omega_{R(A)/R}_{f/R}}{p\Omega_{R(A)/R}_{f/R}} \rightarrow \Omega_{R(B)/R} \rightarrow 0$$

Now, since $R(B)_f \simeq B'_f$, $B'_f$ has only real maximal ideals. Hence, $B'_f \simeq R(C)_f$. Hence,

$$\Omega_{R(B)_f/R} \simeq \Omega_{B'_f/R} \simeq \Omega_{R(C)_f/R}.$$

Hence $\Omega_{R(B)/R}$ is generated by $\chi$, and so the sequence is split exact. Let $s$ be a splitting. Then $(\Omega_{R(A)/R}/p\Omega_{R(A)/R})_f$ is a free $R(B)$-module with a basis $\{ d(a_1), d(a_2), \ldots, d(a_{n-1}), s(\chi) \}$ where for $a \in R(A)$, we denote by $d(a)$, the image of $d(a)$ in $\Omega_{R(A)/R}/p\Omega_{R(A)/R}$. As a consequence, $\wedge_{i=1}^{n-1} da_i \wedge s(\chi) = \rho'$ is a generator for $(R(K)/pR(K))_f$.

Let $\rho \in K$ and $\tau \in L$ be generators of $K_f$ and $L_f$ respectively. Then, $T = \tau \otimes \rho$ is a generator for $E_f$. As above, we can write $D_f \simeq A_f[T, T^{-1}]$ and we can consider
the action of $T$ on $\mathcal{P} = \Theta(\mathcal{M}_0, \omega_{\mathcal{M}_0})$ and $\mathcal{Q} = \Theta(\mathcal{M}_1, \omega_{\mathcal{M}_1})$. By (4.1.13),

$$\text{sign}(T(\mathcal{P})) = \text{sgn}_{(\tau, \rho)}(\mathcal{M}_0, \omega_{\mathcal{M}_0}), \quad \text{sign}(T(\mathcal{Q})) = \text{sgn}_{(\tau, \rho)}(\mathcal{M}_1, \omega_{\mathcal{M}_1}).$$

Since $\Pi : \Psi \to \gamma$ is bijective and $\gamma \subseteq \mathcal{X}_f(\mathbb{R})$ we have $\Psi \subseteq Z_f(\mathbb{R})$. As $T$ defines a continuous and semialgebraic function on $Z_f(\mathbb{R})$, and $\mathcal{P}, \mathcal{Q} \in \Psi$ (which is semialgebraically connected); $\text{sign}(T(\mathcal{P})) = \text{sign}(T(\mathcal{Q}))$. Hence, changing $\rho$ to $-\rho$ if necessary, we may assume without loss of generality that,

$$\text{sgn}_{(\tau, \rho)}(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = 1 = \text{sgn}_{(\tau, \rho)}(\mathcal{M}_1, \omega_{\mathcal{M}_1}).$$

Now, as $\tau \in L$ is a generator of $L_f$ by (3.1.11), there exists $a_n \in \mathbb{R}(A)$ and a surjection

$$\beta : \mathbb{R}(L) \oplus \mathbb{R}(A)^{n-1} \twoheadrightarrow \mathcal{M}_0 \cap \mathcal{M}_1$$

such that

- $(a_1, a_2, \ldots, a_n) = \mathcal{M}_0 \cap \mathcal{M}_1$,
- $\beta(\tau) = a_1 + ha_n$ for some $h \in \mathbb{R}(A)$,
- $\beta(e_i) = a_i; \; 2 \leq i \leq n$ where $(e_2, \ldots, e_n)$ is a basis of $\mathbb{R}(A)^{n-1}$.

Moreover if $\tilde{a}_n$ denotes the image of $a_n$ in $\mathbb{R}(B)(= \mathbb{R}(A)/p\mathbb{R}(A)) \hookrightarrow \mathbb{R}(C)$, then

- $\text{sign}(d(\tilde{a}_n)/\chi) = -1$ at $m_0$ and $\text{sign}(d(\tilde{a}_n)/\chi) = 1$ at $m_1$.

Note that if $d(\tilde{a}_n)/\chi = w \in \mathbb{R}(C) \subset \mathbb{R}(C)_f = \mathbb{R}(B)_f$, then $\wedge_{i=1}^n \tilde{a}_j = w \rho'$.

The surjection $\beta : \mathbb{R}(L) \oplus \mathbb{R}(A)^{n-1} \twoheadrightarrow \mathcal{M}_0 \cap \mathcal{M}_1(= (a_1, a_2, \ldots, a_n))$ gives rise to local orientations $\omega_0$ and $\omega_1$ of $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively as follows:

$$\omega_i : \frac{\mathbb{R}(L)}{\mathcal{M}_i \mathbb{R}(L)} \sim \wedge^n \left( \frac{\mathcal{M}_i}{\mathcal{M}_i^2} \right), \quad \omega_i(\tau'_i) = a_{i1} \wedge a_{i2} \wedge \ldots \wedge a_{in} \quad i = 0, 1$$

where $\tau'_i$ denotes the image of $\tau$ in $\mathbb{R}(L)/\mathcal{M}_i \mathbb{R}(L)$ and $a_{ij}$ denotes the image of $a_j$ in $\mathcal{M}_i/\mathcal{M}_i^2$.

Recall that $\kappa$ and $\tau$ are generators for $\mathbb{R}(L)^2$ and $\mathbb{R}(L)_f$ respectively. Therefore $\kappa = u(\tau \otimes \tau)$ for some $u \in \mathbb{R}(A)^*_f$. Since $\rho$ is a generator of $\mathbb{R}(K)$ and $\rho'$ is a generator of $(\mathbb{R}(K)/p\mathbb{R}(K))_f$, $\tilde{\rho} = \nu \rho'$ for some $\nu \in \mathbb{R}(B)^*_f$ where $\tilde{\rho}$ is the image of $\rho$ in $(\mathbb{R}(K)/p\mathbb{R}(K))_f$. 


Let $u_i, v_i, w_i$ denote images of $u, v, w$ in $\mathbb{R}(A)/\mathcal{M}_i$ respectively. Note that as $\gamma$ is a semialgebraically connected subset of $\text{Spec}(B_f)(\mathbb{R}) \subseteq X_f(\mathbb{R})$ and the points of $X(\mathbb{R})$ corresponding to $\mathcal{M}_0$ and $\mathcal{M}_1$ belong to $\gamma$,

$$\text{sign}(u_0)\text{sign}(u_1) = \text{sign}(v_0)\text{sign}(v_1) = 1.$$ 

On the other hand, by choice of $w$, $\text{sign}(w_0) = -1$ and $\text{sign}(w_1) = 1$. Now using the equalities $\kappa = u(\tau \otimes \tau), \bar{\rho} = \nu \rho'$ and $\wedge^n_{j=1}d\bar{a}_j = wp'$, we see that

$$\text{sgn}_{(\tau,\rho)}(\mathcal{M}_0, \omega_0)\text{sgn}_{(\tau,\rho)}(\mathcal{M}_1, \omega_1) = -1$$

But

$$\text{sgn}_{(\tau,\rho)}(\mathcal{M}_0, \omega_0) = 1 = \text{sgn}_{(\tau,\rho)}(\mathcal{M}_1, \omega_1).$$

Without loss of generality we assume that

$$\text{sgn}_{(\tau,\rho)}(\mathcal{M}_0, \omega_0) = \text{sgn}_{(\tau,\rho)}(\mathcal{M}_0, \omega_1) = 1.$$

$$\text{sgn}_{(\tau,\rho)}(\mathcal{M}_1, \omega_1) = -\text{sgn}_{(\tau,\rho)}(\mathcal{M}_1, \omega_1) = 1.$$

Hence,

$$(\mathcal{M}_0, \omega_0), (\mathcal{M}_1, \omega_1) = (\mathcal{M}_0, \omega_1)$$

in $E(\mathbb{R}(A), \mathbb{R}(L))$.

Since orientations $\omega_0$ and $\omega_1$ on $\mathcal{M}_0$ and $\mathcal{M}_1$ are induced by the surjection $\beta : \mathbb{R}(L) \oplus \mathbb{R}(A)^{n-1} \rightarrow \mathcal{M}_0 \cap \mathcal{M}_1$, we know that in $E(\mathbb{R}(A), \mathbb{R}(L))$,

$$(\mathcal{M}_0, \omega_0) + (\mathcal{M}_1, \omega_1) = 0.$$ 

Hence, in $E(\mathbb{R}(A), \mathbb{R}(L))$,

$$(\mathcal{M}_0, \omega_0) = -(\mathcal{M}_1, \omega_1) = (\mathcal{M}_1, -\omega_1)$$

and so

$$(\mathcal{M}_0, \omega_0) = -(\mathcal{M}_1, \omega_1) = (\mathcal{M}_1, -\omega_1)$$

i.e. $(P) = (Q)$. Thus, the lemma is proved.

We make a few comments about the choice of $f$ in the previous lemma (4.1.15).
Lemma 4.1.16. Let $R$ be a regular domain of dimension $n$ and let $\mathfrak{p}$ be a prime ideal of height $n - 1$ such that the normalisation $S$ of $R/\mathfrak{p}$ is a finite module over $R/\mathfrak{p}$. Let $\mathfrak{c}$ be an ideal of $R$ containing $\mathfrak{p}$ such that $\mathfrak{c}/\mathfrak{p}$ is the conductor ideal of $S$ over $R/\mathfrak{p}$. Let $a_1, a_2, \ldots, a_{n-1} \in R$ such that $(a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p} \cap I$ with $I \not\subseteq \mathfrak{p}$. Let $L$ be a projective $R$-module of rank 1. Then there exists $f \in R \setminus \mathfrak{p}$ such that

1. $f \in \mathfrak{c} \cap (\mathfrak{p} + I)$
2. $L_f \simeq R_f$.

Proof. Since $L_\mathfrak{p} \simeq R_\mathfrak{p}$, there exists $g \in R \setminus \mathfrak{p}$ such that $L_g \simeq R_g$. Since $\mathfrak{p} \not\subseteq \mathfrak{c} \cap (\mathfrak{p} + I)$, there exists $h \in \mathfrak{c} \cap (\mathfrak{p} + I) \setminus \mathfrak{p}$. Now taking $f = gh$, we are through. \(\square\)

Remark 4.1.17. Let $\Psi, \gamma$ be as in the lemma (4.1.15) and $\tilde{\gamma}^{Zar} = \text{Spec}(A/\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of $A$ of height $n - 1$. Since $\mathfrak{R}(A)p_{\mathfrak{R}(A)}$ is regular of dimension $n - 1$, we can always find $a_1, a_2, \ldots, a_{n-1}$ such that $(a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p}R(A) \cap I$ with $I \not\subseteq \mathfrak{p}R(A)$. Then, (4.1.16) shows that there exists $f \in \mathfrak{R}(A) \setminus \mathfrak{p}R(A)$ satisfying conditions (1) and (2) in the hypothesis of (4.1.15). Note that if moreover $\gamma \subset X_f(R) = (\text{Spec}(A_f))(R)$ then for every maximal $\mathcal{M}$ of $\mathfrak{R}(A)$ corresponding to a point of $\gamma$, $(B_1)_{\mathfrak{M}} = \mathfrak{R}(B)\mathfrak{m}$ is a discrete valuation ring where $B_1 = \mathfrak{R}(A)/(a_1, a_2, \ldots, a_{n-1}), \mathfrak{R}(B) = \mathfrak{R}(A)/\mathfrak{p}\mathfrak{R}(A), \overline{\mathcal{M}} = \mathcal{M}B_1$ and $\mathfrak{m} = \mathcal{M}\mathfrak{R}(B)$. In particular every point of $\gamma$ is a smooth point of $\text{Spec}(A/\mathfrak{p})$ (recall $A/\mathfrak{p} = B$). Since an elementary path might contain non smooth points of the curve $\text{Spec}(B)$, it is not always possible to have $f \in \mathfrak{R}(A)$ satisfying condition (3) in the hypothesis of (4.1.15). However, $\text{Spec}(\mathfrak{R}(A)/f) \cap \gamma$ is a finite set, say, $\{Q_1, Q_2, \ldots, Q_t\}$. Since $\gamma$ is totally ordered we can assume that $Q_1 < Q_{i+1}; 1 \leq i \leq t - 1$.

Let $P$ and $Q$ be the initial and end point of $\gamma$ respectively. Then we have $P \leq Q_1 < Q_2 < \ldots < Q_t \leq Q$ and open intervals $[P, Q_1[, ]Q_i, Q_{i+1}[; 1 \leq i \leq t - 1$ and $]Q_t, Q]$ are contained in $X_f(R)$. Since $\Pi|_{\Psi} : \Psi \to \gamma$ is bijective and order preserving we have $\Pi|_{\Psi}^{-1}(P) = \mathcal{P} \leq \Pi|_{\Psi}^{-1}(Q_1) < \Pi|_{\Psi}^{-1}(Q_2) < \ldots < \Pi|_{\Psi}^{-1}(Q_t) \leq \mathcal{Q} = \Pi|_{\Psi}^{-1}(Q)$. Now let $\mathcal{P}', \mathcal{P}'' \in \Psi$ be such that $\Pi|_{\Psi}^{-1}(Q_i) < \mathcal{P}' < \mathcal{P}'' < \Pi|_{\Psi}^{-1}(Q_{i+1})$, then the previous lemma says that $(\mathcal{P}') = (\mathcal{P}'')$ in $E(\mathfrak{R}(A), \mathfrak{R}(L))$. 
The next lemma essentially says that in \( E(\mathbf{R}(A), \mathbf{R}(L)) \)

\[
(\Pi|^{-1}_\psi(Q_i)) = (P') = (P'') = (\Pi|^{-1}_\psi(Q_{i+1})).
\]

**Lemma 4.1.18.** For every point \( P \) of \( Z(\mathbf{R}) \), there exists a semialgebraic neighbourhood \( U_P \) such that if \( P_1, P_2 \in U_P \), then \( \langle P_1 \rangle = \langle P_2 \rangle \).

Before proving this lemma, we state a standard lemma which we will require.

**Lemma 4.1.19.** Let \( A \) be a smooth affine domain of dimension \( n \) over \( \mathbf{R} \) and let \( \mathcal{M} \) be a real maximal ideal of \( A \). Let \( L \) be a rank 1 projective \( \mathcal{A} \)-module. Assume that \( A \) is a surjective image of \( \mathbf{R}^l \) where \( \mathbf{R}^l \) denotes a polynomial algebra in \( l \) variables. Then there exists a set of variables \( \{X_1, \ldots, X_l\} \) (i.e. \( \mathbf{R}^l = \mathbf{R}[X_1, \ldots, X_l] \) and \( f \notin \mathcal{M} \) such that \( A \) is a finite module over \( \mathbf{R}[X_1, \ldots, X_n] \), \( \Omega_{A_f/\mathbf{R}[X_1, \ldots, X_n]} = 0 \) and \( L_f \simeq A_f \).

We prove this lemma in Appendix B. We also discuss this lemma in the context of the semialgebraic implicit function theorem (2.2.5). We now proceed to prove lemma (4.1.18).

**Proof.** Let \( \Theta(\mathcal{M}, \omega_\mathcal{M}) = \mathcal{P} \). Let \( P \in X(\mathbf{R}) \) be the point corresponding to \( \mathcal{M} \), i.e. \( \Pi(P) = P \). Since \( A \) is affine, we can assume that \( \text{Spec}(A) \) is a closed \( n \)-dimensional subvariety of the affine space \( \mathbb{A}_\mathbf{R}^l \). Hence, \( X(\mathbf{R}) \) is a closed algebraic subset of \( \mathbb{A}_\mathbf{R}^l \). Then, by (4.1.19), there exists a suitable choice of a coordinate system of \( \mathbb{A}_\mathbf{R}^l \), such that the projection map \( \pi : \mathbb{A}_\mathbf{R}^l \to \mathbb{R}^n \) when restricted to \( X(\mathbf{R}) \) has finite fibers. Moreover, there exists \( f \in A \) such that \( f \notin \mathcal{M} \), \( L_f \simeq A_f \) and \( \Omega_{A_f/\mathbf{R}} \) is generated by \( dX_1, \ldots, dX_n \). Therefore, by the semialgebraic implicit function theorem (2.2.5), there exists a semialgebraic Euclidean neighbourhood \( U \) of \( P \) contained in the Zariski neighbourhood \( \text{Spec}(A_f) \) such that the restriction of the projection map \( \pi \) to \( U \) is a Nash isomorphism onto an open ball \( \mathcal{B} \) in \( \mathbb{R}^n \) with center \( \pi(P) \).

We fix an element \( \tau \in L \) such that it generates \( L_f \). Note that \( K_f \) is generated by \( \rho = \bigwedge_{i=1}^n dX_i \). Note further that since \( f \notin \mathcal{M} \), \( \mathcal{P} \in Z_f(\mathbf{R}) \). Let \( D_f = A_f[T, T^{-1}], T = \tau \otimes \rho \). Then,

\[
Z_f(\mathbf{R}) = X_f(\mathbf{R}) \times \mathbb{R}^+ \bigcup X_f(\mathbf{R}) \times \mathbb{R}^-.
\]
Without loss of generality we may assume that $\text{sgn}_{(\tau,\rho)}(\mathcal{P})$ is positive, i.e. 

$$\mathcal{P} \in U \times \mathbb{R}^+ \subseteq X_f(\mathbb{R}) \times \mathbb{R}^+.$$ 

Let $U_{\mathcal{P}} = U \times \mathbb{R}^+$. By (2.1.5), $(\Theta^{-1}(\mathcal{M}, T - 1)) = (\Theta^{-1}(\mathcal{M}, uT - 1))$ in $E(\mathbb{R}(A), \mathbb{R}(L))$ for $u > 0$. So we may assume that $\mathcal{P}$ corresponds to $(\mathcal{M}, T - 1)$. Again by (2.1.5), to prove the proposition it is enough to prove that, $(\Theta^{-1}(\mathcal{M}, T - 1)) = (\Theta^{-1}(\mathcal{M}', T - 1))$ in $E(\mathbb{R}(A), \mathbb{R}(L))$ for every $\mathcal{M}' \subseteq U$. Let $\mathcal{P}'$ be the element of $Z(\mathbb{R})$ corresponding to $(\mathcal{M}', T - 1)$.

**Case 1.** Suppose $\Pi(\mathcal{P}') = \Pi(\mathcal{P}) = P$. Then $\mathcal{P}' = \mathcal{P}$ and so $(\mathcal{P}) = (\mathcal{P}')$ in $E(\mathbb{R}(A), \mathbb{R}(L))$.

**Case 2.** Suppose $\Pi(\mathcal{P}') = P' \neq P$. Let $W$ be a semialgebraic open subset of $\mathbb{R}^l$ such that $W \cap X(\mathbb{R}) = U$. Since $P \neq P'$, $\pi(P) \neq \pi(P')$. Without loss of generality we assume that $\pi(P) = (0, \ldots, 0)$ and $\pi(P') = (\delta_1, \ldots, \delta_n)$. Moreover, without loss of generality, we assume that $\delta_n > 0$. The line $\mathcal{L}$ joining the two points $\pi(P) = (0, \ldots, 0)$ and $\pi(P') = (\delta_1, \ldots, \delta_n)$ is given by $n - 1$ equations:

$$H_i : X_i - \zeta_iX_n, \quad \zeta_i = \delta_i/\delta_n, \quad 1 \leq i \leq n - 1.$$ 

Let $\mathcal{L}_1$ be the segment of $\mathcal{L}$ contained in $\pi(U) = B$. Then there exists an open interval $(a, b) \supset [0, \delta_n]$ and a Nash isomorphism from $(a, b)$ to $\mathcal{L}_1$ given by:

$$t \mapsto (\zeta_1t, \zeta_2t, \ldots, \zeta_{n-1}t, t).$$

Composing the above function with $(\pi_{|\mathcal{L}_1})^{-1}$ we obtain a Nash embedding $F(t) = (f_1(t), \ldots, f_n(t), f_j(t))$ from $(a, b)$ to $(\pi_{|\mathcal{L}_1})^{-1}(\mathcal{L}_1) \subseteq U \subseteq R^l$. It is easy to see that $f_i(t) = \zeta_it$ for $i = 1, \ldots, n - 1$, $f_n(t) = t$ and $f_j(t) = g_j(\zeta_i t, \ldots, \zeta_{n-1} t, t)$ for $j = n + 1, \ldots, l$. Note that $F(0) = P$ and $F(\delta_n) = P'$.

Let $\gamma = F([0, \delta_n])$. Then, $\gamma \subseteq U \subseteq \text{Spec}(A_f)$. Moreover, since $H_i$ vanishes at every point of $\gamma$ for $1 \leq i \leq n - 1$, $\gamma \subseteq (\text{Spec}((A/(H_1, H_2, \ldots, H_{n-1}))f)(\mathbb{R})$. Note that $\Omega_{A_f/R}[X_1, X_2, \ldots, X_n] = 0; R[X_1, X_2, \ldots, X_n]/(H_1, H_2, \ldots, H_{n-1}) \simeq \mathbb{R}[T]$. Therefore, $(A/(H_1, H_2, \ldots, H_{n-1}))f)$ is a regular ring. Hence onward we simply write $(H_1, H_2, \ldots, H_{n-1})$ for $(H_1, H_2, \ldots, H_{n-1})f)$.

Consider

$$(H_1, H_2, \ldots, H_{n-1}) = (\cap_{i=1}^2 p_i) \cap (\cap_{j=r_1+1}^2 p_j) \cap (\cap_{k=1}^3 q_k),$$
the primary decomposition of \((H_1, H_2, \ldots, H_{n-1})\) in \(A\) where
\[
f \notin \bigcup_{i=1}^{r_1} p_i, f \in \bigcap_{j=r_1+1}^{r_2} p_j
\]
and \(q_k\) are primary but not prime ideals, which, as \((A/(H_1, H_2, H_{n-1}))_f\) is regular implies that
\[
f^m \in \bigcap_{k=1}^{r_3} q_k, m \in \mathbb{N}.
\]
Hence, \((H_1, H_2, \ldots, H_{n-1})_f = \bigcap_{i=1}^{r_1} (p_i)_f\). Then,
\[
\text{Spec}(A/(H_1, H_2, \ldots, H_{n-1}))_f = \bigcup_{i=1}^{r_1} \text{Spec}(A/(p_i))_f.
\]
Since \(\gamma \subset \text{Spec}(A/(H_1, H_2, \ldots, H_{n-1}))_f = \bigcup_{i=1}^{r_1} \text{Spec}(A/(p_i))_f\), we have
\[
\gamma = \bigcup_{i=1}^{r_1} \gamma \cap (\text{Spec}(A/(p_i))_f)(\mathbb{R}).
\]
Since \(\gamma \cap \text{Spec}(A/(p_i)_f)\) is a closed semialgebraic subset of \(\gamma\) and \(\gamma\) is semialgebraically connected, there exists \(i\) such that \(\gamma = \gamma \cap \text{Spec}(A/p_i)\) while the other intersections are empty. Without loss of generality let \(i = 1\) and let us denote \(p_1 = p\). Hence, \(\gamma \subset \text{Spec}(A/p)\) and hence, \(\gamma^\text{ Zar} = \text{Spec}(A/p)\). \(\gamma\) inherits a natural order from \([0, \delta_n]\) and every point of \(\gamma\) is a smooth point of \(\text{Spec}(A/(H_1, H_2, \ldots, H_{n-1}))\) as well as of \(\text{Spec}(A/p)\). Hence, \(\gamma\) is an elementary path with \(\Pi(\mathcal{P}) = P = F(0)\) as the initial point and \(\Pi(\mathcal{P'}) = P' = F(\delta_n)\) as the endpoint.

Consider the map \(D_f \to A_f\) given by \(T \mapsto 1\). This induces a section
\[
s : \text{Spec}(A_f) \to \text{Spec}(D_f), \quad \mathcal{M} \mapsto (\mathcal{M}, T - 1), \mathcal{M} \in \text{Max}(A_f).
\]
Then let \(\Psi = s(\gamma)\). Then clearly \(\Psi\) is an elementary path in \(Z(\mathbb{R})\) with starting point \(\mathcal{P}\) and endpoint \(\mathcal{P}'\). Note that since it is a section, \(\Pi(\Psi) = \gamma\) and the map \(\Pi|_{\Psi} : \Psi \to \gamma\) is a bijection. Let
\[
I = ((\cap_{i=2}^{r_2} p_i) \bigcap (\cap_{j=r_1+1}^{r_2} p_j) \bigcap (\cap_{k=1}^{r_3} q_k))\mathbb{R}(A).
\]
Then
\[
(H_1, H_2, \ldots, H_{n-1}) = p\mathbb{R}(A) \cap I.
\]
Let \( B = A/p \) and \( B_1 = A/(H_1, H_2, \ldots, H_{n-1}) \). Then since \( B_f \) and \( (B_1)_f \) are regular, \( f \in \sqrt{\mathfrak{C} \cap (pR(A) + I)} \) where \( \mathfrak{C} \) is the conductor ideal of \( R(A) \) with respect to its normalisation. By choice of \( f \) and \( \gamma, L_f \approx A_f \approx K_f \) and \( \operatorname{Spec}(R(A)/f) \cap \gamma = \emptyset \). Therefore, we can apply (4.1.15) and hence we get that \( (P) = (P') \). \( \square \)

**Proposition 4.1.20.** Let \( \Psi \subset Z(R) \) be an elementary path and let \( \Pi(\Psi) = \gamma \).
Suppose \( \Pi|_{\Psi} : \Psi \to \gamma \) is bijective. Let \( P \) and \( Q \) be the initial and end points of \( \Psi \). Then \( (P) = (Q) \) in \( \operatorname{E}(R(A), R(L)) \).

*Proof.* If \( \Psi \) is singleton then there is nothing to prove. So we assume that \( \Psi \) is nondegenerate. Then, \( \gamma \) is also a nondegenerate elementary path in \( X(R) \).

Let \( \tilde{\Psi}_{\text{Zar}} = \operatorname{Spec}(D/q) \) and \( \tilde{\nu}_{\text{Zar}} = \operatorname{Spec}(A/p) \). Then \( p = A \cap q \) and \( p \) is a prime ideal of height \( n - 1 \) of \( A \). Since \( R(A) \) is regular ; there exist \( a_1, a_2, \ldots, a_{n-1} \in R(A) \) such that \( (a_1, a_2, \ldots, a_{n-1}) = pR(A) \cap I \) with \( I \not\in pR(A) \). Let \( B' \) be the normalisation of \( R(B) = (R(A)/pR(A)) \) and let \( \mathfrak{C} \) be an ideal of \( R(A) \) containing \( pR(A) \) such that \( \mathfrak{C}/pR(A) = \mathfrak{c}/R(B) \), the conductor ideal of \( B' \) over \( R(B) \).

Then, by (4.1.16), there exists \( f \in R(A) \setminus p \) such that \( f \in \mathfrak{C} \cap (pR(A) + I) \) and \( R(L)_f \approx R(A)_f \). Let \( \Upsilon = \{ T \in X(R)|pR(A) + (f) \subseteq \mathcal{M}_T \} \) where \( \mathcal{M}_T \) denotes the maximal ideal of \( R(A) \) corresponding to \( T \) and \( \Upsilon' = \Upsilon \cap \gamma \). Since \( f \not\in pR(A) \) (an ideal of height \( n - 1 \), \( \Upsilon \) and hence \( \Upsilon' \) are finite sets.

If \( \Upsilon' = \emptyset \) then by lemma (4.1.15), \( (P) = (Q) \) in \( \operatorname{E}(R(A), R(L)) \).

So we assume \( \Upsilon' \neq \emptyset \). Let \( \Upsilon_1 = \Pi^{-1}(\Upsilon') \cap \Psi \). Then as \( \Pi|_{\Psi} : \Psi \to \gamma \) is bijective, \( \Upsilon_1 \) is a finite subset of \( \Psi \), say \( \Upsilon_1 = \{ Q_1, Q_2, \ldots, Q_i \} \). Since \( \Psi \) is totally ordered, without loss of generality we assume that \( Q_i < Q_{i+1}; 1 \leq i \leq t - 1 \). Let \( P = Q_0 \) and \( Q = Q_{t+1} \). Then we have \( Q_0 \leq Q_1 < Q_2 < \ldots < Q_{t} \leq Q_{t+1} \).

Consider an interval \( [Q_i, Q_{i+1}], 1 \leq i \leq t - 1 \). Then, by (4.1.18), there exists \( U_{Q_i} \subset Z(R) \) such that for any two points \( S, S' \) in \( U_{Q_i} \), we have \( (S) = (S') \). Note that by [17, Proposition 7.5] \( Q_i \) is contained in the closure of \( [Q_i, Q_{i+1}] \). Therefore, \( U_{Q_i} \cap Q_i, Q_{i+1} \neq \emptyset \) (and hence is infinite). Choose \( S_{i,1} \in U_{Q_i} \cap Q_i, Q_{i+1} \). Similarly, we can choose \( S_{i+1,0} \in U_{Q_{i+1}} \cap Q_i, Q_{i+1} \). Then, \( [S_{i,1}, S_{i+1,0}] \) is a sub-interval of \( [Q_i, Q_{i+1}] \). Consider \( \Psi_i = [S_{i,1}, S_{i+1,0}] \). Then note that \( \tilde{\Psi}_i \) is an infinite closed subset of the irreducible curve \( \operatorname{Spec}(D/q) \) and hence has to equal it. Hence, \( \tilde{\Psi}_i \) = \( \operatorname{Spec}(D/q) \) and \( \Psi_i \) is actually an elementary path. Further,
\[ \pi(\Psi_i) \cap T = \emptyset. \] Then, by (4.1.15), \((S_{i,1}) = (S_{t+1,0}). \) Since \(S_{t,0}, S_{t,1} \in U_Q, \) we also have \((S_{t,0}) = (S_{t,1}). \) Hence, we get that \((S_{1,1}) = (S_{t,0}). \) If \(Q_0 = Q_1, \) let \(S_{0,1} = S_{1,1}, \) else consider \([S_{0,1}, S_{1,0}]. \) Similarly, if \(Q_t = Q_{t+1}, \) let \(S_{t,1} = S_{t+1,1}, \) else consider \([S_{t,1}, S_{t+1,0}]. \) In all four cases, we get \((S_{0,1}) = (S_{t+1,0}). \) But since \(S_{0,1}, Q_0 \in U_P \) and \(S_{t+1,0}, Q_{t+1} \in U_Q, \) we have \((S_{0,1}) = (Q_0) \) and \((Q_{t+1}) = (S_{t+1,0}). \) Hence, we get \((Q_0) = (Q_{t+1}) \) i.e. \((M_0, \omega_{M_0}) = (M_1, \omega_{M_1}) \) in \(E(R(A), R(L)). \)

Finally, we prove the main result of this chapter.

**Theorem 4.1.21.** Let \(A, L, K, E, D, X(R), Z(R), Y, \Theta : Y \to Z(R) \) be as in the beginning of this chapter. Let \(\mathcal{P} = \Theta((M_0, \omega_{M_0})) \) and \(Q = \Theta((M_1, \omega_{M_1})) \) be two distinct points of \(Z(R) \) lying in the same semialgebraically connected component of \(Z(R). \) Then \((M_0, \omega_{M_0}) = (M_1, \omega_{M_1}) \) in \(E(R(A), R(L)). \)

**Proof.** Since \(\mathcal{P} \) and \(Q \) lie in the same component of \(Z(R), \) we can join \(\mathcal{P} \) and \(Q \) by a semialgebraic path \(\Psi. \) Then by (2.3.7), this path breaks into finitely many non-degenerate elementary paths \(\Psi_i, 1 \leq i \leq r \) such that \(\Psi_i \cap \Psi_{i+1} = \{S_i\} \) and \(S_i \) is the starting point of \(\Psi_{i+1} \) and the endpoint of \(\Psi_i. \) Moreover, \(\mathcal{P} = S_0 \) is the starting point of \(\Psi_1 \) and \(Q = S_r \) is the end point of \(\Psi_r. \) Therefore it is enough to show that \((S_i) = (S_{i+1}), 0 \leq i \leq r - 1 \) in \(E(R(A), R(L)). \)

Hence we can assume without loss of generality that \(\Psi \) is a non-degenerate elementary path in \(Z(R), \) with initial point \(\mathcal{P} \) and end point \(Q. \) If \(\Pi(\Psi) \) (where \(\Pi : Z(R) \to X(R)) \) is singleton, then by (4.1.14) we are through. So we assume that \(\Pi(\Psi) \) is not singleton (and hence infinite).

In this case, by [12, Theorem 3.1], there exists a sub-division \(\mathcal{P} = \mathcal{P}_0 < \mathcal{P}_1 < \ldots < \mathcal{P}_t = Q \) such that if \(\Psi_j = [\mathcal{P}_j, \mathcal{P}_{j+1}] \), then \(\Pi|_{\Psi_j} : [\mathcal{P}_j, \mathcal{P}_{j+1}] \to \Pi(\Psi_j) \) is order-preserving and bijective. Therefore, by (4.1.15) \((\mathcal{P}_j) = (\mathcal{P}_{j+1}) \) in \(E(R(A), R(L)) \) for \(0 \leq j \leq t - 1. \) Therefore \((\mathcal{P}) = (Q) \) in \(E(R(A), R(L)). \) \qed
Chapter 5

Structure theorem for $E(\mathbb{R}(A), \mathbb{R}(L))$

In this chapter, we prove the main theorem, namely the structure theorem Theorem A.

We recall the setup once again. Let $X = \text{Spec}(A)$ be a smooth affine variety over $\mathbb{R}$ of dimension $n \geq 2$. Assume further that the set $X(\mathbb{R})$ of real points is not empty, hence infinite. Let $L$ be a projective $A$-module of rank 1. We denote $K_A = \wedge^n(\Omega_A/\mathbb{R})$ by $K$.

Let $\mathcal{E} = L \otimes_A K$. Let $D = \oplus_{-\infty < i < \infty} \mathcal{E}^i$. Let $Z = \text{Spec}(D)$. Then there is a natural map $A \hookrightarrow D$ which gives rise to a natural surjection $Z \twoheadrightarrow X$ which induces a natural map $Z(\mathbb{R}) \twoheadrightarrow X(\mathbb{R})$ which we denote by $\Pi$. Looked at in the Euclidean topology, this gives an $\mathbb{R}^*$-bundle over $X(\mathbb{R})$.

Let

$$Y = \{(\mathcal{M}, \omega_\mathcal{M}) | \mathcal{M} \in X(\mathbb{R}), \ \omega_\mathcal{M} : \frac{L}{\mathcal{M}L} \sim \wedge^n(\frac{\mathcal{M}}{\mathcal{M}^2})\}.$$  

Then there is a natural bijection $\Theta : Y \sim Z(\mathbb{R})$. Recall that $E(\mathbb{R}(A), \mathbb{R}(L))$ is a quotient of the free abelian group on the set $Y$.

Let $C_1, C_2, \ldots, C_r, C_{r+1}, \ldots, C_t$ be the closed and bounded components of $X(\mathbb{R})$.

Note that $L, K, E = L \otimes_A K$ correspond to semialgebraic line bundles on $X(\mathbb{R})$. Let $L_i, K_i, E_i$ be the restrictions of these line bundles to $C_i$. Then $\Pi^{-1}(C_i)$ is the complement of the zero section of $E_i$. Note that $L_i$ is isomorphic to $K_i$ as a semialgebraic line bundle (denoted by $L_i \simeq K_i$) if and only if $E_i$ is a semialgebraically trivial line bundle over $C_i$. 

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Now suppose that \( L_i \simeq K_i \) for \( 1 \leq i \leq r \) and \( L_i \not\simeq K_i \) for \( r + 1 \leq i \leq t \).

**Lemma 5.1.22.** \( 2(\mathcal{M}, \omega_\mathcal{M}) = 0 \) in \( E(\mathbb{R}(A), \mathbb{R}(L)) \) where the point corresponding to \( \mathcal{M} \) lies in \( C_i \), \( r + 1 \leq i \leq t \).

**Proof.** Since \( \Pi : Z(\mathbb{R}) \to X(\mathbb{R}) \) is a continuous semi-algebraic map, every component of \( Z(\mathbb{R}) \) is contained in \( \Pi^{-1}(C) \) for some component \( C \) of \( X(\mathbb{R}) \). In particular, if \( \Pi^{-1}(C) \) is semi-algebraically connected then \( \Pi^{-1}(C) \) is a semi-algebraically connected component of \( Z(\mathbb{R}) \). Note that by (2.2.8), if \( C \) is a closed and bounded component of \( X(\mathbb{R}) \) then \( \Pi^{-1}(C) \) has two components if and only if \( E \vert_C \) is trivial, otherwise \( \Pi^{-1}(C) \) is semi-algebraically connected.

Since \( E_i \) is not trivial for \( r + 1 \leq i \leq t \), by (2.2.8), \( \Pi^{-1}(C_i) \) is semi-algebraically connected and hence is a component of \( Z(\mathbb{R}) \). Now if \( \mathcal{M} \) is a maximal ideal of \( \mathbb{R}(A) \) such that the point corresponding to it lies in \( C_i \) then \( \Theta((\mathcal{M}, \omega_\mathcal{M})) \) and \( \Theta((\mathcal{M}, -\omega_\mathcal{M})) \in \Pi^{-1}(C_i) \). Since \( \Pi^{-1}(C_i) \) is semi-algebraically connected for \( r + 1 \leq i \leq t \), by (4.1.21), \( (\mathcal{M}, \omega_\mathcal{M}) = (\mathcal{M}, -\omega_\mathcal{M}) \) in \( E(\mathbb{R}(A), \mathbb{R}(L)) \). Therefore, by (2.2.9), \( 2(\mathcal{M}, \omega_\mathcal{M}) = 0 \) in \( E(\mathbb{R}(A), \mathbb{R}(L)) \). \( \square \)

**Lemma 5.1.23.** Let \( \mathcal{M} \) be a maximal ideal of \( \mathbb{R}(A) \) corresponding to a point \( T' \) in \( C \), where \( C \) is an unbounded component of \( X(\mathbb{R}) \). Let \( \omega_\mathcal{M} \) be a local \( L \)-orientation of \( \mathcal{M} \). Then \( (\mathcal{M}, \omega_\mathcal{M}) = 0 \) in \( E(\mathbb{R}(A), \mathbb{R}(L)) \).

**Proof.** Let \( \tilde{X} \) be the smooth projective completion of \( X = \text{Spec}(A) \). Then there exists an affine open subset \( X_1 = \text{Spec}(A_1) \) of \( \tilde{X} \) such that \( X_1(\mathbb{R}) = \tilde{X}(\mathbb{R}) \). Then if \( X' = X \cap X_1 \), we have \( X'(\mathbb{R}) \). Let \( A' \) be the coordinate ring of \( X' \). Since \( X \cap X_1 \) is an affine open subset of \( X_1 \) and \( \text{Pic}(\mathbb{R}(A_1)) \) is a 2-torsion group, \( \mathbb{R}(A_1) \) is a localization of \( \mathbb{R}(A) \). Now since \( X \cap X_1 \) is an open subset of \( X \) and \( X'(\mathbb{R}) = X(\mathbb{R}) \), we have \( X'(\mathbb{R}) \). Let \( L_1 \) be a rank 1 projective over \( A_1 \) such that \( L_1 \) and \( L \) define the same projective module over \( \mathbb{R}(A) \). Note that since \( \tilde{X} \) is projective, \( X_1(\mathbb{R}) = \tilde{X}(\mathbb{R}) \) is closed and bounded.

Since \( X'(\mathbb{R}) = X(\mathbb{R}) \), we can regard \( C \) as a semi-algebraically connected subset of \( X_1(\mathbb{R}) \). Therefore there exists a component \( \tilde{C} \) of \( X_1(\mathbb{R}) \) such that \( C \subset \tilde{C} \). Since \( \tilde{C} \) is closed and bounded and \( C \) is not closed and bounded, there exists \( T \in \tilde{C} \) such that \( T \not\subset C \). Note that \( \tilde{C} \not\subset X(\mathbb{R}) \) (otherwise \( \tilde{C} \) being semi-algebraically connected, \( \tilde{C} \subset C \)). Therefore we can assume that \( T \not\subset X(\mathbb{R}) \).
Let $\mathcal{M}_T$ denote the corresponding maximal ideal of $R(A_1)$. Since $T \notin X(R)$, we have $\mathcal{M}_T R(A) = R(A)$. Let $\omega_{\mathcal{M}_T}$ be a local $L_1$-orientation of $\mathcal{M}_T$. Since $T, T' \in \tilde{C}$, (4.1.21) implies that either

$$(\mathcal{M}_T, \omega_{\mathcal{M}_T}) = (\mathcal{M}, \omega_{\mathcal{M}})\lor (\mathcal{M}_T, -\omega_{\mathcal{M}})$$

in $E(R(A_1), R(L_1))$. Since $R(A) = R(A')$ is a localization of $R(A_1)$, there exists a (surjective) group homomorphism from $E(R(A_1), R(L_1))$ to $E(R(A), R(L))$. Since under this group homomorphism, $(\mathcal{M}_T, \omega_{\mathcal{M}_T}) \mapsto 0$ in $E(R(A), R(L))$, $(\mathcal{M}, \omega_{\mathcal{M}}) = 0$ in $E(R(A), R(L))$. \hfill $\Box$

Let $T_i \in C_i$, $1 \leq i \leq t$ and let $\mathcal{M}_i$ be the maximal ideal of $R(A)$ corresponding to $T_i$. Let $\omega_i$ be a local $L$-orientation of $\mathcal{M}_i$ for $1 \leq i \leq t$. Let $F$ be a free abelian group with a basis $(e_1, \ldots, e_t)$. Let $\Delta : F \to E(R(A), R(L))$ be a group homomorphism defined by $\Delta(e_i) = (\mathcal{M}_i, \omega_i)$; $1 \leq i \leq t$.

**Proposition 5.1.24.** $\Delta$ is surjective. As a consequence $E_0(R(A), R(L))$ is a vector space of rank $\leq t$ over $\mathbb{Z}/(2)$.

**Proof.** Since $E(R(A), R(L))$ is generated by $Y$, it is enough to show that all elements of $Y$ are in the image of $\Delta$. Let $(\mathcal{M}, \omega_{\mathcal{M}}) \in Y$. Suppose the point corresponding to $\mathcal{M}$ does not lie in $C_i$ for any $i, 1 \leq i \leq t$. Then, by (5.1.23), $(\mathcal{M}, \omega_{\mathcal{M}}) = 0$ and hence, it trivially lies in the image of $\Delta$. If the point corresponding to $\mathcal{M}$ lies in $C_i$ for some $i, 1 \leq i \leq t$, then, using (4.1.21), $(\mathcal{M}, \omega_{\mathcal{M}}) = (\mathcal{M}_i, \omega_i)$ or $(\mathcal{M}, \omega_{\mathcal{M}}) = (\mathcal{M}_i, -\omega_i) = - (\mathcal{M}_i, \omega_i)$. Hence, $(\mathcal{M}, \omega_{\mathcal{M}}) = \Delta(e_i)$ or $(\mathcal{M}, \omega_{\mathcal{M}}) = -\Delta(e_i)$. Therefore, $\Delta$ is surjective.

By (2.2.9), $E_0(R(A), R(L))$ is a vector space over $\mathbb{Z}/(2)$ and $E(R(A), R(L))$ maps surjectively onto it. Hence, it is a vector space of rank $\leq t$ over $\mathbb{Z}/(2)$. \hfill $\Box$

**Remark 5.1.25.** By (5.1.22), the map $\Delta$ induces the surjection :

$$\bigoplus_{i=1}^t \mathbb{Z}e_i \bigoplus \bigoplus_{i=t+1}^t (\mathbb{Z}/2)e_i \xrightarrow{\Delta} E(R(A), R(L)).$$

We will prove that $\bar{\Delta}$ is an isomorphism.
Recall that there is a bijection $\Theta : Y \xrightarrow{\sim} Z(\mathbb{R})$. Note that for $1 \leq i \leq r$, there is a section $C_i \to \Pi^{-1}(C_i)$ which induces the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{R}^* & \xleftarrow{p_2} & C_i \times \mathbb{R}^* \xrightarrow{s_i} Z(\mathbb{R})|_{C_i} \\
& \downarrow{p_1} & \downarrow{\Pi} \\
& & C_i
\end{array}
\]

Then, we have a map $Z(\mathbb{R}) \to \{-1, 0, 1\}$ sending

\[p \mapsto \begin{cases} 
0 & \text{if } \Pi(p) \notin C_i \\
-1 & \text{if } \Pi(p) \in C_i \text{ and } p_2(s_i(p)) < 0 \\
1 & \text{if } \Pi(p) \in C_i \text{ and } p_2(s_i(p)) > 0
\end{cases}\]

Consider the composite map $\text{sign}_i : Y \to \{-1, 0, 1\}, 1 \leq i \leq r$ and consider the induced map on the free abelian group on $Y$ to $\mathbb{Z}$. Note that if $(\mathcal{M}, \omega_0)$ and $(\mathcal{M}, \omega_1)$ are two local orientations of $\mathcal{M}$ (where $\mathcal{M}$ corresponds to a point in $C_i$), then there exists $\lambda \in \mathbb{R}^*$ such that $\omega_0 = \lambda \omega_1$. Then, $p_2(s_i(\Theta(\mathcal{M}, \omega_0))) = \lambda p_2(s_i(\Theta(\mathcal{M}, \omega_1)))$ and so, $\text{sign}_i((\mathcal{M}, \omega_0)) = \text{sign}(\lambda)\text{sign}_i((\mathcal{M}, \omega_1)), 1 \leq i \leq r$.

Recall that $E(\mathbb{R}(A), \mathbb{R}(L))$ is a quotient of the free abelian group on $Y$. The next lemma shows that $\text{sign}_i$ in fact factors through $E(\mathbb{R}(A), \mathbb{R}(L))$.

**Lemma 5.1.26.** Let $I$ be a finite intersection of maximal ideals of $\mathbb{R}(A)$ and let $\beta : \mathbb{R}(L) \oplus \mathbb{R}(A)^{n-1} \to I$ be a surjection. Let $1 \leq i \leq r$ and let $\mathcal{M}'_1, \ldots, \mathcal{M}'_l$ be all the maximal ideals of $\mathbb{R}(A)$ such that:

1. $I \subseteq \mathcal{M}'_j, 1 \leq j \leq l$.
2. the point corresponding to $\mathcal{M}'_j$ is contained in $C_i, 1 \leq j \leq l$.

Let $\omega_j$ be the local $\mathbb{R}(L)$-orientation of $\mathcal{M}'_j$ induced by $\beta$. Then

\[\sum_{j=1}^{l} \text{sign}_i((\mathcal{M}'_j, \omega_j)) = 0.\]

**Proof.** Let $\{f_2, f_3, \ldots, f_n\}$ be a basis of $\mathbb{R}(A)^{n-1}$ and let $\beta(f_k) = a_k, 2 \leq k \leq n$. We can assume that if $J = (\beta(\mathbb{R}(L)), a_2, \ldots, a_{n-1})$ then $J$ is a prime ideal of height $n - 1$ and $\text{Spec}(A/J)$ is a smooth irreducible curve [22, Theorems 1.3 and...
1.4]. Let "tilde" denote reduction modulo the ideal \( J \). Then we have the following exact sequence

\[
0 \longrightarrow \frac{J}{J^2} \longrightarrow \frac{\Omega_{R(\hat{A})/R}}{J\Omega_{R(\hat{A})/R}} \longrightarrow \frac{\Omega_{R(\hat{A})/R}}{\Omega_{R(\hat{A})/R}} \longrightarrow 0.
\]

Since \( R(\hat{A}) \) is smooth, we can apply (2.3.2) and hence, \( \Omega_{R(\hat{A})/R} \) is a free \( R(\hat{A}) \)-module of rank one. Let \( \theta \) be a generator of \( \Omega_{R(\hat{A})/R} \) and let \( v \) be its preimage in \( \Omega_{R(\hat{A})/R} / J(\Omega_{R(\hat{A})/R}) \). Let \( F = R(L) \oplus R(A)^{n-2} \). Then, since \( \beta(F) = J \) we get an isomorphism \( \tilde{\beta} : F/JF \simeq J/J^2 \). Hence \( \Omega_{R(\hat{A})/R} / J\Omega_{R(\hat{A})/R} \simeq F/JF \oplus \Omega_{R(\hat{A})/R} \).

Since \( \wedge^{n-1}(F) \simeq R(L) \) and \( \wedge^n(\Omega_{R(\hat{A})/R}) = R(K) \), we get that

\[
\frac{R(L)}{JR(L)} \simeq \frac{\wedge^n(F \oplus \Omega_{R(\hat{A})/R})}{JF} \simeq \frac{\wedge^n\left( \frac{J}{J^2} \oplus \Omega_{R(\hat{A})/R} \right)}{\frac{J}{J\Omega_{R(\hat{A})/R}}} \cong \frac{R(K)}{JR(K)}
\]

\[
\tilde{l} \mapsto (\tilde{l} \wedge (\wedge_{i=2}^{n-1} \tilde{a}_i)) \otimes \theta \mapsto (\tilde{\beta}(l) \wedge (\wedge_{i=2}^{n-1} \tilde{a}_i)) \otimes \theta \mapsto d(\tilde{\beta}(l)) \wedge (\wedge_{i=2}^{n-1} \tilde{d}a_i) \wedge v.
\]

This induces a natural map

\[
\frac{R(A)}{JR(A)} \simeq \frac{R(L)}{JR(L)} \otimes \frac{R(L)}{JR(L)} \simeq \frac{R(\mathcal{E})}{JR(\mathcal{E})}.
\]

Let \( \chi \in R(\mathcal{E}) \) be such that \( 1 \mapsto \tilde{\kappa} \mapsto \tilde{\chi} \) under the above map. This induces a map \( R(D)/JR(D) \rightarrow R(A)/JR(A) \) mapping \( \tilde{\chi} \mapsto 1 \). This gives another section of the line bundle restricted to \( (V(J))(R) \subset X(R) \). Let "bar" denote reduction modulo \( M'_j \). Let \( \omega_j \) be the orientation sending \( \tilde{l} \mapsto \tilde{\beta}(l) \wedge (\wedge_{i=2}^{n-1} \tilde{a}_i) \wedge d^{-1}_{M'_j} (\tilde{v}) \) for each \( j \). Then, \( \Theta(M'_j, \omega_j) = (M'_j, \chi - 1) \) for each \( j \).

Note that \( \tilde{l} = (\tilde{a}_n) \). Since \( \theta \) is a generator of \( \Omega_{R(\hat{A})/R} \), we have \( d(\tilde{a}_n) = u \theta \) for some \( u \in \overline{R(A)} \). Note that \( d(\tilde{a}_n) \) is non-zero in \( \Omega_{R(\hat{A})/R}/M'_j \Omega_{R(\hat{A})/R} \) and hence is a generator. Hence, \( u \) is a unit modulo \( M'_j \), i.e \( u_j = u(M'_j) \in R^* \). Therefore,

\[
\overline{d(\tilde{\beta}(l))} \wedge (\wedge_{i=2}^{n-1} \tilde{d}a_i) \wedge \tilde{v} = u_j^{-1} \overline{d(\tilde{\beta}(l))} \wedge (\wedge_{i=2}^{n} \tilde{d}a_i).
\]

Hence,

\[
\overline{\beta}(l) \wedge (\wedge_{i=2}^{n-1} \tilde{a}_i) \wedge d^{-1}_{M'_j} (\tilde{v}) = u_j^{-1} \overline{\beta}(l) \wedge (\wedge_{i=2}^{n} \tilde{a}_i).
\]

Hence, \( \omega_j = u_j^{-1} \omega_j \) for every \( j \). Hence, \( sign_i(M'_j, \omega_j) = sign(u_j) sign_i(M'_j, \omega_j) \) and so, \( sign(u_j) sign_i(M'_j, \omega_j) = sign_i(M'_j, \omega_j) \).
Since \((V(J))(R) \subset X(R)\), any component of \((V(J))(R)\) is disjoint from \(C_i\) or completely contained in it. Hence, \(C_i \cap (V(J))(R) = \bigcup_k W_{ik}\) where \(W_{ik}\) are components of \((V(J))(R)\). Then, we have

\[
\sum_{j=1}^l \text{sign}_i((\mathcal{M}'_j, \omega_j)) = \sum_{k} \sum_{j : \mathcal{M}'_j \in W_{ik}} \text{sign}_i((\mathcal{M}'_j, \omega_j)).
\]

Hence, it is enough to show that for each \(W_{ik}\), \(\sum_{j : \mathcal{M}'_j \in W_{ik}} \text{sign}_i((\mathcal{M}'_j, \omega_j)) = 0\).

We note that \(W_{ik}\) are closed subsets of \(C_i\) and hence are closed and bounded components of \((V(J))(R)\) and hence components of its “completion”. Fix \(W_{ik} = W\). Since \(W\) is a closed and bounded component of \((V(J))(R)\) and hence a component of its “completion”, by [16, Theorem 3.3], we get that

\[
\sum_{j : \mathcal{M}'_j \in W} \text{sign}(\frac{d(a_n)}{\theta}(\mathcal{M}'_j)) = \sum_{j : \mathcal{M}'_j \in W} \text{sign}(u_j) = 0.
\]

Now,

\[
\sum_{j : \mathcal{M}'_j \in W} \text{sign}_i((\mathcal{M}'_j, \omega_j)) = \sum_{j : \mathcal{M}'_j \in W} \text{sign}(u_j)\text{sign}_i((\mathcal{M}'_j, \bar{\omega}_j)).
\]

By definition, \(\text{sign}_i((\mathcal{M}'_j, \bar{\omega}_j)) = \text{sign}(p_2(s_i(\Theta((\mathcal{M}'_j, \bar{\omega}_j)))))\). Since \(\Theta((\mathcal{M}'_j, \bar{\omega}_j)) = (\mathcal{M}, \chi - 1)\), the section induced by \(\chi\) on \((V(J))(R)\) is given by the map \(\mathcal{M} \mapsto \Theta((\mathcal{M}'_j, \omega)) = (\mathcal{M}, \chi - 1)\). Hence, \(p_2(s_i(\Theta((\mathcal{M}'_j, \omega))))\) is a continuous, semi-algebraic map on \(W\) which is semi-algebraically connected. Hence, \(p_2(s_i(\mathcal{M}, \chi - 1))\) has the same sign for all \(\mathcal{M} \in W\) and so,

\[
\sum_{j : \mathcal{M}'_j \in W} \text{sign}(u_j)\text{sign}_i((\mathcal{M}'_j, \omega)) = \pm \sum_{j : \mathcal{M}'_j \in W} \text{sign}(u_j) = 0.
\]

Hence, the lemma is proved.$\square$

Thus, the map \(\text{sign}_i\) factors through \(E(R(A), R(L))\). Without loss of generality we can assume that for the chosen generators \((\mathcal{M}_i, \omega_i)\) of \(E(R(A), R(L))\), \(\text{sign}_i((\mathcal{M}_i, \omega_i)) = 1\) for \(1 \leq i \leq r\). We now complete the proof that \(\bar{\Delta}\) is an isomorphism and hence, obtain the structure theorem for \(E(R(A), R(L))\). To achieve this, we exploit the relation between the Euler class group and the Chow
group, and use a theorem of Colliot-Thélène and Scheiderer. We set up some
notation for the same.

Recall that there exists a natural surjection \( \Gamma_L : E(A, L) \twoheadrightarrow E(R(A), R(L)) \).
Then, the following lemma is generalised from [6, Lemma 5.6], where it is proved
in the case that \( L = A \). Note that here we are viewing the Euler class groups as
defined for a commutative, Noetherian ring.

**Lemma 5.1.27.** Every element of \( \ker (\Gamma_L) \) is of the form \( (I, \omega_I) \) where \( I \) is an
ideal of height \( n \) not contained in any real maximal ideal and \( \omega_I \) is a local \( L- 
orientation.

**Proof.** Suppose \( (J, \omega_J) \) is a nonzero element of \( \ker (\Gamma_L) \). Then, \( (JR(A), \omega_J \otimes_A R(A)) = 0 \) which means by [6, Theorem 4.2] that \( \omega_J \otimes_A R(A) \) is a global
orientation, i.e. there exists \( \beta' : R(L) \oplus R(A)^{n-1} \to JR(A) \). Then, there exists
\( f \in A \) which does not lie in any real maximal ideal, such that \( \beta : L_f \oplus A^{n-1}_f \to J_f \).
Then, by [6, Theorem 2.14], we can choose an ideal \( J_1 \) in \( A \) of height \( n \) such that
it is comaximal with \( J \) and \( (f) \) and \( (J, \omega_J) + (J_1, \omega_{J_1}) = 0 \) in \( E(A, L) \). Then, the
image of \( (J_1, \omega_{J_1}) \) is 0 in \( E(A_f, L_f) \). Once again, by [6, Theorem 4.2], we get that
there exists \( \alpha : L_f \oplus A^{n-1}_f \to (J_1)_f \) which restricts to \( \omega_{J_1} \otimes_A A_f \) modulo \( J_1 \).

Restrict \( \alpha \) to \( L \oplus A^{n-1} \). Then, we can multiply the map \( \alpha \) by a suitably
high power of \( f \) so that this restriction actually maps into \( J_1 \). Further, we can
choose this power to be even, i.e. \( f^{2k} \alpha \) is a surjection from \( L_f \oplus A^{n-1}_f \to (J_1)_f \)
which when restricted to \( L \oplus A^{n-1} \) maps it into \( J_1 \). Further, going modulo \( J_1 \), \( f \)
is a unit, hence, \( (J_1, \omega_{J_1}) = (J_1, f^{2kn}\omega_{J_1}) \) by 2.1.5. Let \( K_1 \) denote the image of
\( L \oplus A^{n-1} \) under \( f^{2k} \alpha \).

Now, a careful analysis of the proof of [1, Proposition 3.1] shows that there
exists an element \( \sigma \in Aut(L_f \oplus A^{n-1}_f) \) such that it is a product of transvections
such that \( f^{2k} \alpha \circ \sigma \) when restricted to \( L \oplus A^{n-1} \) maps onto an ideal \( K \subseteq A \)
which has height \( \geq n \). Let us denote by \( \omega_K \) the local \( L \)-orientation induced by
\( f^{2k} \alpha \circ \sigma \). Note that since \( \sigma \) is a product of transvections, modulo \( J_1 \), it yields
an elementary matrix and hence, the local orientation induced by \( f^{2k} \alpha \circ \sigma \) is
elementarily equivalent to the local orientation induced by $f^{2k\alpha}$, i.e.
\[
\begin{array}{c c c}
  L_f & \xrightarrow{f^{2k\alpha}} & \bar{J}_f \\
  J_{f_f} & \xrightarrow{\tilde{\sigma}} & J_{f_f} \\
  \downarrow & & \downarrow \\
  L_f & \xrightarrow{f^{2k\alpha}} & \bar{J}_f \\
  J_{f_f} & & J_{f_f}
\end{array}
\]
and $\tilde{\sigma} \in E_n(A_f/J_{f_f}) = E_n(A/J_1)$ and hence, $(J_1, \omega_{J_1}) = (J_1, \tilde{f}^{2k\alpha} \circ \sigma)$

Then, this means that $K \subseteq J_1$, $K_f = (J_1)_f$, $(K, \omega_K) = 0$ in $E(A, L)$ and $(K_f, \omega_{K_f} \otimes_A A_f) = (J_{f_f}, \omega_{J_{f_f}} \otimes_A A_f) = 0$ in $E(A_f, L_f)$. From these relations and the fact that $J_1$ is comaximal with $f$, we can conclude that $K = J_1 \cap I$ where $I$ is an ideal containing some power of $f$ and which is comaximal with $J_1$ ($J_1$ is precisely the intersection of the primary components of $K$ which do not contain $f$ and $I$ is the intersection of those which do contain $f$). Hence, denoting the local $L$-orientation
\[
(f^{2k\alpha} \circ \sigma) \otimes_A (A/I)^{n-1} \rightarrow K \\
\]
by $\omega_I$, we get that $0 = (K, \omega_K) = (J_1, \omega_{J_1}) + (I, \omega_I)$ in $E(A, L)$. Hence, $(I, \omega_I) = (J, \omega_J)$ and since $I$ contains a power of $f$ which is not contained in any real maximal ideal, neither is $I$. This completes the proof. \qed

We denote $\ker (\Gamma_L)$ by $E^R(L)$ where $\overline{R}$ denotes the algebraic closure of $R$. We first show that $E^R(L)$ is a torsion-free and divisible group, and hence $E(A, L) \simeq E(R(A), R(L)) \oplus E^R(L)$.

Let $A_{\overline{R}} = A \otimes_R \overline{R}$ and $Y = \text{Spec}(A_{\overline{R}})$. Then $A_{\overline{R}}$ is a smooth affine $\overline{R}$-algebra of dimension $n \geq 2$ and hence $CH_0(Y)$ is a divisible group. Let $\pi : Y \rightarrow X$ be the canonical map. Note that $\pi$ is a finite morphism and it induces group homomorphisms $\pi^* : CH_0(X) \rightarrow CH_0(Y)$ and $\pi_* : CH_0(Y) \rightarrow CH_0(X)$ such that the composition $\pi_* \pi^*$ is multiplication by 2. Let $G = \pi_*(CH_0(Y))$.

**Lemma 5.1.28.** $G = \pi_*(CH_0(Y))$ is a divisible group and if $n \geq 2$ then it is torsion-free.

**Proof.** Since $G$ is a surjective image of the divisible group $CH_0(Y)$, $G$ is divisible. Let $n \geq 2$. Then, by [9, Proposition 2.1], $CH_0(Y)$ is torsion-free. Now let
$z \in \text{CH}_0(X)$ be a torsion element, say, $r z = 0$. Then, $\pi^*(r z) = r \pi^*(z) = 0$. But $\pi^*(z) \in \text{CH}_0(Y)$ and $\text{CH}_0(Y)$ is torsion-free, which implies that $\pi^*(z) = 0$. Therefore, $\pi_* \pi^*(z) = 2 z = 0$. So any torsion element of $\text{CH}_0(X)$ is 2-torsion and hence every torsion element of $G$ is 2-torsion. But since $G$ is divisible, $G$ must then be torsion-free. \(\square\)

**Proposition 5.1.29.** $(\Theta_L)|_{(E^R(L))}$ is an isomorphism with image $G$.

**Proof.** $G$ is generated by $\pi_* [\mathcal{N}]$ where $\mathcal{N}$ is a maximal ideal of $A_R$ and $[\mathcal{N}]$ denotes the zero cycle in $\text{CH}_0(Y)$ associated to $A_R/\mathcal{N}$. Let $\mathcal{M} = A \cap \mathcal{N}$. Note that if $\mathcal{M}$ is a complex maximal ideal of $A$ then $[\mathcal{M}] = \pi_* [\mathcal{N}]$ and if $\mathcal{M}$ is a real maximal ideal then $2 [\mathcal{M}] = \pi_* [\mathcal{N}]$ ($[\mathcal{M}]$ denotes the zero cycle associated to $A/\mathcal{M}$ in $\text{CH}_0(X)$). This shows that if $I$ is an ideal of $A$ of height $n$ such that $I/I^2$ is generated by $n$ elements and $I$ is contained in only complex maximal ideals then $[I] \in G$.

We now use the second definition of the Euler class group, i.e. the definition for a commutative, Noetherian ring. Using this definition, by (5.1.27), every element of $E^R(L)$ is of the form $(I, \omega_I)$, where $I$ is an ideal of height $n$ (with $I/I^2$ generated by $n$ elements) contained only in complex maximal ideals (notice that we do not make any claims about the $I$ being a radical ideal) and $\omega_I$ is a local $L$-orientation of $I$, we see that $\Theta_L(E^R(L)) \subset G$.

Now to complete the proof it is enough to show that if $\mathcal{M}$ is a real maximal ideal of $A$ then $2 [\mathcal{M}] \in \Theta_L(E^R(L))$. Let $\omega_\mathcal{M}$ be a local $L$-orientation of $\mathcal{M}$. Then $2 [\mathcal{M}] = \Theta_L((\mathcal{M}, \omega_\mathcal{M}) + (\mathcal{M}, -\omega_\mathcal{M}))$. By (2.2.9), $(\mathcal{M}, \omega_\mathcal{M}) + (\mathcal{M}, -\omega_\mathcal{M}) \in E^R(L)$. Hence we are through.

The assertion that $2 [\mathcal{M}] \in G$ still holds even in the case dimension of $A$ is 1 as $\mathcal{M}^2 R(A)$ is a principal ideal. \(\square\)

Let $\Psi_L = (\Theta_L)|_{(E^R(L))}$. Then, using the above lemmas, we get the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & E^R(L) & \rightarrow & E(A, L) & \rightarrow & E(R(A), R(L)) & \rightarrow & 0 \\
\Psi_L & \downarrow & \Theta_L & \downarrow & \Phi_L & & & & \\
0 & \rightarrow & G & \rightarrow & \text{CH}_0(X) & \rightarrow & \text{CH}_0(X)/G & \rightarrow & 0 \\
\end{array}
$$

(*)
We now state a result which tells us the structure of the group $\text{CH}_0(X)/G$.

**Theorem 5.1.30.** [11, Theorem 1.3(d)] Assume that $X(\mathbb{R})$ has precisely $t$ closed and bounded components. Then, $\text{CH}_0(X)/G$ is a vector space of dimension $t$ over the field $\mathbb{Z}/(2)$ generated by any point of the component.

**Theorem 5.1.31.**

$$
\bigoplus_{i=1}^r \mathbb{Z}e_i \bigoplus \bigoplus_{i=r+1}^t (\mathbb{Z}/2)e_i \rightarrow E(\mathbb{R}(A), \mathbb{R}(L))
$$

is an isomorphism.

**Proof.** We recall that there is a natural map from $E(\mathbb{R}(A), \mathbb{R}(L)) \rightarrow \text{CH}_0(X)/G$ where $G = \pi_*(\text{CH}_0(X))$ and from (5.1.30), $\text{CH}_0(X)/G \simeq (\mathbb{Z}/2)^t$ where every point of any component is a generator. Hence, for each $i : r+1 \leq i \leq t$, there is a natural surjection $E(\mathbb{R}(A), \mathbb{R}(L)) \rightarrow \mathbb{Z}/(2)$ obtained by first taking the surjection $E(\mathbb{R}(A), \mathbb{R}(L)) \rightarrow \text{CH}_0(X)/G(\simeq (\mathbb{Z}/2)^t)$ followed by the projection to the $i$th factor of $(\mathbb{Z}/2)^t$. We denote it by $\text{sign}_i$. Then, putting together the earlier maps $\text{sign}_i$ for $1 \leq i \leq r$ (defined on page 58) and the above maps $\text{sign}_i$, $r+1 \leq i \leq t$, we get a map

$$
\Delta' : E(\mathbb{R}(A), \mathbb{R}(L)) \rightarrow \bigoplus_{i=1}^r \mathbb{Z}e_i \bigoplus \bigoplus_{i=r+1}^t (\mathbb{Z}/2)e_i.
$$

sending $(M, \omega_M) \mapsto \sum_{i=1}^t \text{sign}_i((M, \omega_M))e_i$. Since $\Delta' \circ \bar{\Delta} = id$, we get that $\bar{\Delta}$ is an isomorphism. This finishes the proof. \hfill \Box

This proves the main structure theorem (1.4) which we state again below.

**Theorem A.** Let $\mathbb{R}$ be a real closed field. Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the $\mathbb{R}$-rational points of the variety. Let $K$ denote the module $\wedge^n(\Omega^2_{A/\mathbb{R}})$. Let $P$ be a projective $A$-module of rank $n$ and let $\wedge^n(P) = L$. Let $C_i, 1 \leq i \leq t$ be the closed and bounded semialgebraically connected semialgebraic components of $X(\mathbb{R})$. Let $L_i$ and $K_i$ be the restriction of the semialgebraic line bundles corresponding to $L$ and $K$ respectively, to $C_i$. Let $L_i \simeq K_i$, for $1 \leq i \leq r$ and $L_i \not\simeq K_i$, for $r+1 \leq i \leq t$. Let
\[x_i \in C_i \text{ and let } M_i \text{ be the corresponding maximal ideal of } R(A). \text{ Let } \omega_i \text{ be a local } R(L)\text{-orientation of } M_i. \text{ Then, }\]

\[
\bigoplus_{i=1}^t \mathbb{Z}e_i \bigoplus \bigoplus_{i=r+1}^t (\mathbb{Z}/2)e_i \cong \text{E}(R(A), R(L))\]

sending \(e_i \mapsto (M_i, \omega_i)\) is an isomorphism.

Using this, we finally prove a theorem which allows us to classify when the vanishing of the top Chern class is sufficient to split off a free summand.

**Corollary 5.1.32.**

\[E_0(R(A), R(L)) \cong \text{CH}_0(X)/G\]

and hence

\[E_0(A, L) \rightarrow \text{CH}_0(X)\]

is an isomorphism.

**Proof.** We know that \(E_0(R(A), R(L))\) is a vector space of rank \(\leq t\) from the above structure theorem **Theorem A** and (2.2.9). But \(E_0(R(A), R(L)) \rightarrow \text{CH}_0(X)/G\) and by (5.1.30), \(\text{CH}_0(X)/G\) is a vector space of rank \(t\). Hence, so is \(E_0(R(A), R(L))\) and

\[E_0(R(A), R(L)) \cong \text{CH}_0(X)/G.\]

So using this relation and (5.1.29) in the diagram (*), and denoting the kernel of \(E_0(A, L) \rightarrow E_0(R(A), R(L))\) by \(E^R_0(L)\), we get an induced diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & E^R_0(L) & \rightarrow & E_0(A, L) & \rightarrow & E_0(R(A), R(L)) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & G & \rightarrow & \text{CH}_0(X) & \rightarrow & \text{CH}_0(X)/G & \rightarrow & 0
\end{array}
\]

(**)

Therefore, using the 5-lemma, we get \(E_0(A, L) \cong \text{CH}_0(X).\) \(\square\)

**Theorem B.** Let \(R\) be a real closed field. Let \(X = \text{Spec}(A)\) be a smooth affine variety of dimension \(n \geq 2\) over \(R\). Let \(X(R)\) denote the \(R\)-rational points of the variety. Let \(K\) denote the module \(\wedge^n(\Omega_{A/R})\). Let \(P\) be a projective \(A\)-module of rank \(n\) and let \(\wedge^n(P) = L\). Assume that \(c_n(P) = 0\) in \(\text{CH}_0(X)\). Then \(P \cong A \oplus Q\) in the following cases:
1. $X(\mathbf{R})$ has no closed and bounded semialgebraically connected component.

2. For every closed and bounded semialgebraically connected component $W$ of $X(\mathbf{R})$, $L_W \not\simeq K_W$ where $K_W$ and $L_W$ denote restriction of (induced) line bundles on $X(\mathbf{R})$ to $W$.

3. $n$ is odd.

Moreover, if $n$ is even and $L$ is a rank 1 projective $A$-module such that there exists a closed and bounded semialgebraically connected component $W$ of $X(\mathbf{R})$ with the property that $L_W \simeq K_W$, then there exists a projective $A$-module $P$ of rank $n$ such that $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ (hence $c_n(P) = 0$) but $P$ does not have a free summand of rank 1.

Proof. We note that due to (5.1.29), the diagram (*) can be re-written as:

\[
\begin{array}{cccc}
\ker(\Theta_L) & \overset{\sim}{\longrightarrow} & \ker(\Phi_L) \\
\downarrow & & \downarrow \\
0 \longrightarrow \mathbb{E}^R(L) \longrightarrow \mathbb{E}(A, L) \longrightarrow \mathbb{E}(\mathbb{R}(A), \mathbb{R}(L)) \longrightarrow 0 \\
\Psi_L \downarrow & & \theta_L \downarrow & & \Phi_L \downarrow \\
0 \longrightarrow G \longrightarrow \text{CH}_0(X) \longrightarrow \text{CH}_0(X)/G \longrightarrow 0
\end{array}
\]

Hence, using (5.1.30) and the structure theorem (1.4), we get that $\ker(\Phi_L)$ ($\simeq \ker(\Theta_L)$) is a free abelian group of rank $r$ where $r$ denotes the number of closed and bounded components $C_i$ of $X(\mathbf{R})$ with the property that $L_{C_i} \simeq K_{C_i}$.

Let $P$ be a projective $A$-module of rank $n$ with $\Lambda^n(P) \simeq L$ and let $\chi : L \rightarrow \Lambda^n(P)$ be an $L$-orientation of $P$. Then, $\Theta_L(e(P, \chi)) = c_n(P)$. In view of (2.1.3), to prove the theorem, it is enough to prove that $c_n(P) = 0 \Rightarrow e(P, \chi) = 0$.

Proof in cases 1. and 2.

Note that in these cases, by the structure theorem of $\mathbb{E}(\mathbb{R}(A), \mathbb{R}(L))$, $\Phi_L$ is an isomorphism and hence, $\Theta_L$ is an isomorphism.

Proof in case 3. When $n$ is odd, there is an automorphism $\Delta$ of $P$ with determinant $-1$. Let $\alpha : P \rightarrow I$ where $I$ is a finite intersection of maximal ideals. Let $\omega_I$ be a local $L$-orientation of $I$ induced by $\alpha$. Using this and $\Delta$, we get that $2e(P, \chi) = (I, \omega_I) + (I, -\omega_I)$. Since the canonical map $\text{E}_0(A, L) \rightarrow \text{CH}_0(X)$ is an isomorphism by (5.1.32) and $(I) \mapsto c_n(P)$, we have $(I) = 0$ in $\text{E}_0(A, L)$. 

\[
\begin{array}{cccc}
\ker(\Theta_L) & \overset{\sim}{\longrightarrow} & \ker(\Phi_L) \\
\downarrow & & \downarrow \\
0 \longrightarrow \mathbb{E}^R(L) \longrightarrow \mathbb{E}(A, L) \longrightarrow \mathbb{E}(\mathbb{R}(A), \mathbb{R}(L)) \longrightarrow 0 \\
\Psi_L \downarrow & & \theta_L \downarrow & & \Phi_L \downarrow \\
0 \longrightarrow G \longrightarrow \text{CH}_0(X) \longrightarrow \text{CH}_0(X)/G \longrightarrow 0
\end{array}
\]
Therefore, as in [2, Proposition 3.7], \(2e(P, \chi) = (I, \omega_I) + (I, -\omega_I) = 0\). Since \(c_n(P) = 0\), \(e(P, \chi) \in \ker(\Theta_L)\), which is a free abelian group. Hence, \(e(P, \chi) = 0\).

Finally, let \(n\) be even, and \(L_W \simeq K_W\) for some closed and bounded semialgebraically connected component \(W\). Then, using the structure theorem (1.4), \(\ker(\Theta_L) \simeq \ker(\Phi_L) \neq 0\). Since \(E_0(A, L) \to \CH_0(X)\) is an isomorphism, by abuse of notation, we denote the canonical map \(E(A, L) \to E_0(A, L)\) by \(\Theta_L\). Then, as in [5, Lemma 3.3], there exists a reduced ideal \(J\) of height \(n\) such that \(J\) is a surjective image of \(L \oplus A^{n-1}\) and a local \(L\)-orientation \(\omega_J\) which is not a global orientation, i.e. \((J, \omega_J) \neq 0\) in \(E(A, L)\). Since \(n\) is even, as in [5, Lemma 3.6], we can get a rank \(n\) projective module \(P\), which is stably isomorphic to \(L \oplus A^{n-1}\) (i.e. \(P \oplus A \simeq L \oplus A^{n-1} \oplus A\)) and \(\chi : L \to \wedge^n(P)\) such that \(e(P, \chi) = (I, \omega_I) \neq 0\) in \(E(A, L)\). Note that since \(P\) is stably isomorphic to \(L \oplus A^{n-1}\), \(c_n(P) = 0\) but \(e(P, \chi) \neq 0\) and hence, by (2.1.3), \(P \not\simeq Q \oplus A\). This completes the proof. \(\Box\)
CHAPTER 5. STRUCTURE THEOREM FOR $E(R(A), R(L))$
Appendix A

Real Closed Extensions

Let $R \hookrightarrow R'$ be an extension of real closed fields and let $X = \text{Spec}(A)$ be an affine variety over $R$ and let $L$ be a rank 1 projective $A$-module. Let $A' = A \otimes_R R'$, $X' = \text{Spec}(A')$ and $L' = L \otimes_A A'$. Then in this appendix we show that $E(A, L) \to E(A', L')$ and $E(R(A), R(L)) \to E(R'(A'), R'(L'))$ are both injective. If $X$ is also smooth, then $E(R(A), R(L)) \to E(R'(A'), R'(L'))$ is actually a canonical isomorphism as a result of the structure theorem we have proved 1.4.

This program was carried out in [7] in the case $R' = \mathbb{R}$. We maintain the above notations throughout this appendix.

Remark A.1.1. Note that for a real maximal ideal $m$ of $A$, $mA'$ is a real maximal ideal of $A'$ (but the first quotient is $R$ while the second is $R'$). Hence, fixing a closed embedding of $X(R)$ in $R^l$ (for suitable $l$), we can regard the topological space $X(R)$ as a subset of $X'(R')$. Let $W_1', W_2', \ldots, W_s'$ be the components of $X'(R')$. Let $W_i = W_i' \cap X(R)$. Then, $W_1, W_2, \ldots, W_s$ are precisely the components of $X(R)$ (for a proof of a more general result see [10, Proposition 5.3.6]). Further, $W_i'$ is closed and bounded if and only if $W_i$ is closed and bounded.

We state the Artin-Lang homomorphism theorem ([10, Thm. 4.1.2]).

Theorem A.1.2. Let $A$ be a finite type $R$-algebra. If there exists an $R$-algebra homomorphism $\phi : A \to R'$ into a real closed extension $R'$ of $R$, then there exists an $R$-algebra homomorphism $\psi : A \to R$.

In particular, if $A$ is an $R$-subalgebra of $R'$, then we get a retraction from $A$ to $R$. 

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Let \( k \hookrightarrow k' \) be a field extension. Let \( X = \text{Spec}(A) \) be an affine variety of dimension \( n \geq 2 \) over \( k \) and let \( A' = A \otimes_k k' \), \( X' = \text{Spec}(A') \). Let \( \Gamma \) be the indexing set for all finite type algebras \( C_\alpha \) over \( k \) contained in \( k' \). Then \( \varprojlim_{\alpha \in \Gamma} C_\alpha = k' \) and hence, \( A' = \varprojlim_{\alpha \in \Gamma} A_\alpha \) where \( A_\alpha = A \otimes_k C_\alpha \).

Let \( L \) be a projective \( A \)-module of rank 1 and \( L' = L \otimes_A A' \). Since \( A' \) is a ring of dimension \( n \) which is faithfully flat over \( A \), it is easy to see that there exist canonical maps \( E(A, L) \to E(A', L') \) and \( E_0(A, L) \to E_0(A', L') \). Moreover, if \( k \) and \( k' \) are real closed fields, then we can extend these maps canonically to maps \( E(k(A), k(L)) \to E(k'(A'), k'(L')) \) and \( E_0(k(A), k(L)) \to E_0(k'(A'), k'(L')) \).

Now, we state a lemma, the proof of which is very easy and hence, we only outline the proof.

**Lemma A.1.3.** Let \( \{ A_\alpha | \alpha \in \Gamma \} \) be a direct system of rings and \( A = \varprojlim_{\alpha \in \Gamma} A_\alpha \).

1. Let \( M \) be a finitely presented \( A \)-module. Then, there exists \( \alpha \in \Gamma \) and a finitely presented \( A_\alpha \)-module \( M_\alpha \) such that \( M \cong M_\alpha \otimes_{A_\alpha} A \).

2. Let \( f : M \to N \) be a homomorphism between two finitely presented \( A \)-modules. Then, there exists \( \alpha \in \Gamma \), finitely presented \( A_\alpha \)-modules \( M_\alpha \), \( N_\alpha \), and an \( A_\alpha \)-homomorphism \( f_\alpha : M_\alpha \to N_\alpha \) such that \( M \cong M_\alpha \otimes_{A_\alpha} A \), \( N \cong N_\alpha \otimes_{A_\alpha} A \) and

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
M_\alpha \otimes_{A_\alpha} A & \xrightarrow{f_\alpha} & N_\alpha \otimes_{A_\alpha} A
\end{array}
\]

In particular, if \( f \) is a surjection, then we can choose \( f_\alpha \) to be a surjection.

**Proof.** Since \( M \) is a finitely presented module, there is an exact sequence
\( A^n \xrightarrow{h} A^m \to M \to 0 \). Then the map \( h \) is given by an \( m \times n \) matrix \( H \) with coefficients in \( A \). Since \( A = \varprojlim_{\alpha \in \Gamma} A_\alpha \), \( \exists \alpha \in \Gamma \) such that the entries of \( H \) come from \( A_\alpha \), i.e. there is a matrix \( H_\alpha \) with entries in \( A_\alpha \) which map to the corresponding entries of the matrix \( H \). Let \( h_\alpha : A_\alpha^n \to A_\alpha^m \) be the corresponding map. Let \( M_\alpha \) be the cokernel of the map given by \( A_\alpha^n \xrightarrow{h_\alpha} A_\alpha^m \). Then, \( M_\alpha \) is finitely presented and \( M \cong M_\alpha \otimes_{A_\alpha} A \). This proves the first part. For the second part, let \( g \) be the composite map \( A^m \to M \xrightarrow{f} N \). Choose \( M_\beta \) and \( N_\beta \) as in the first part. Then, one can define a map \( g_\beta : A_\beta^m \to N_\beta \) so that \( g_\beta \otimes_{A_\beta} A = g \). Then,
the composite map \(g_\beta \circ h_\beta : A^n_\beta \to N_\beta\) is such that \(g_\beta \circ h_\beta \otimes_{A_\beta} A = g \circ h = 0\). Hence, by the property of the direct limit, and since \(n\) is finite, there exists \(\alpha \succ \beta\) such that \(g_\alpha \circ h_\alpha = (g_\beta \otimes_{A_\beta} A_\alpha) \circ (h_\beta \otimes_{A_\beta} A_\alpha) = g_\beta \circ h_\beta \otimes_{A_\beta} A_\alpha = 0\). Since \(M_\alpha = M_\beta \otimes_{A_\beta} A_\alpha\) is the cokernel of \(h_\alpha : A^n_\alpha \to A^n_\alpha\), there is an induced map \(f_\alpha : M_\alpha \to N_\alpha\). Clearly, this map satisfies the required property. Further, if \(f\) was a surjection, then so is \(g\) and so one can choose \(g_\beta\) to be a surjection which will make \(g_\alpha\) and hence \(f_\alpha\) a surjection. Hence, proved. \(\Box\)

We are now in a position to prove the main proposition of this appendix.

**Proposition A.1.4.** The canonical maps \(E(R(A), R(L)) \to E(R'(A'), R'(L'))\) and \(E(A, L) \to E(A', L')\) are injective. Further, if \(X\) is smooth, then \(E(R(A), R(L)) \to E(R'(A'), R'(L'))\) is an isomorphism.

**Proof.** (i) Injectivity: Let \(J\) be an ideal of height \(n\) in \(R(A)\) such that \(J/J^2\) is generated by \(n\) elements and let \(\omega_J\) be a surjective map from \(F = R(L) \oplus R(A)^{n-1} \to J/J^2\) (a local \(L\)-orientation). Let \(I = JR'(A')\). Now suppose there is a surjective map \(\theta_I : F \otimes_{R(A)} R'(A') \to I\) which is a lift of \(\omega_J \otimes_{R(A)} R'(A')\). Then, by (A.1.3), there exists an affine \(R\)-subalgebra \(C\) of \(R'\) such that if \(T = \{1 + \sum_{i=1}^{l} f_i^2 | f_1, f_2, \ldots, f_l \in A \otimes_{R} C\}\), and \(D = T^{-1}(A \otimes_{R} C)\), then there is a surjection \(\theta : F \otimes_{R(A)} A \to JD\), which is a lift of \(\omega_J \otimes_{R(A)} D\).

Since \(R \hookrightarrow C \hookrightarrow R'\), by (A.1.2), \(R\) is a retract of \(C\) and hence, \(A\) is a retract of \(A \otimes_{R} C\). Let \(\beta : A \otimes_{R} C \to A\) be an \(A\)-algebra homomorphism which gives rise to this retract. Let \(S_A = \{1 + \sum_{i=1}^{p} g_i^2 | g_1, g_2, \ldots, g_p \in A\}\). Since \(S_A \subseteq T\) and \(\beta(T) = S_A\), \(R(A) \subseteq D\) and \(\beta\) induces a retraction \(\sigma : D \to R(A)\). In view of this, it is easy to see that the map \(\theta \otimes_{D} R(A)\) is indeed a lift of \(\omega_J\). Thus, the map \(E(R(A), R(L)) \to E(R'(A'), R'(L'))\) is injective.

Virtually the same proof works for \(E(A, L) \to E(A', L')\).

(ii) Isomorphism: Now assume that \(X\) is smooth. Let \(W_1', \ldots, W_t'\) be the closed and bounded semialgebraically connected components of \(X'(R')\) in the Euclidean topology. Denote by \(L_i'\) and \(K_i'\) the restriction to \(W_i'\) of the semialgebraic line bundle corresponding to \(L'\) and \(K' = \Omega_{A'/R'}\) respectively. Assume that \(L_i' \cong K_i'\), for \(1 \leq i \leq r\) and \(L_i' \not\cong K_i'\), for \(r + 1 \leq i \leq t\).
Then by the structure theorem (1.4) applied over \( R' \), we get that

\[
\oplus_{i=1}^{r} \mathbb{Z} \bigoplus \oplus_{i=r+1}^{t}(\mathbb{Z}/2) \sim E(R'(A'), R'(L'))
\]

Further, given any maximal ideal \( m'_i \subset R'(A') \) such that the point corresponding to it lies in \( W'_i \) and any \( R'(L') \)-orientation \( \omega'_i \), \( E(R'(A'), R'(L')) \) is generated by \( (m'_i, \omega'_i) \) for \( 1 \leq i \leq t \).

Note that by (A.1.1), if \( W' \) is a connected component of \( X'(R') \), then \( W' \cap X(R) \neq \emptyset \). Hence, there exists a maximal ideal \( m_i \) of \( R(A) \), such that \( m_i \in W_i = W'_i \cap X(R) \). Now we choose any \( R(L) \)-orientation \( \omega_i \) of \( m_i \). Once again applying the structure theorem (1.4), we get that

\[
\oplus_{i=1}^{r} \mathbb{Z} \bigoplus \oplus_{i=r+1}^{t}(\mathbb{Z}/2) \sim E(R(A), R(L))
\]

and once again \( E(R(A), R(L)) \) is generated by \( (m_i, \omega_i) \). But then letting \( m'_i = m_i R'(A') \) and the orientation \( \omega'_i \) to be the one obtained from \( \omega_i \), it is clear that

\[
E(R(A), R(L)) \sim E(R'(A'), R'(L')).
\]

\( \square \)
Appendix B

Noether Normalisation

We state a slightly general version of (4.1.19), give a brief idea of its proof and then show how it allows us to use the implicit function theorem.

Lemma B.1.1. Let A be a smooth affine domain of dimension n over R and let M be a real maximal ideal of A. Let L be a rank 1 projective A-module. Assume that A is a surjective image of R[Y_1, Y_2, \ldots, Y_l]. Then there exists a set of variables \{X_1, \ldots, X_l\} (i.e. R[Y_1, Y_2, \ldots, Y_l] = R[X_1, \ldots, X_l]) and f \notin M such that, if we denote by x_i the image of X_i in A,

- A is a finite module over R[x_1, \ldots, x_n]
- M_f = (x_1, x_2, \ldots, x_n)_f
- \Omega_{A_f/R[x_1, x_2, \ldots, x_n]} = 0
- L_f \simeq A_f

Proof. Throughout the proof, we use capital letters for variables and small letters for their images in A. For example, y_i denotes the image of Y_i in A.

We note that since M is a real maximal ideal of A which is a surjective image of \( R[Y_1, Y_2, \ldots, Y_l] \), it must be of the form \((y_1 - \alpha_1, y_2 - \alpha_2, \ldots, y_l - \alpha_l)\)A where \( \alpha_i \in \mathbb{R} \). First, we make a linear change of coordinates (which we call \( Y'_i \)) so that \( M \) corresponds to the origin. Further, since \( A \) is smooth, \( M/M^2 \) is generated by \( n \) elements as a vector space over \( \mathbb{R} \). Then, since \( \{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_l\} \) is a generating set, w.l.g. we can assume that \( \{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n\} \) forms a basis of \( M/M^2 \).
Now, this means that $\bar{y}_i = \sum_{j=1}^n \beta_j y_j$ where $\beta_j \in \mathbb{R}$. Change the variables further to the set

$$Z_i = \begin{cases} Y'_i & 1 \leq i \leq n \\ Z_i = Y'_i - \sum_{j=1}^n \beta_j Y'_j & n + 1 \leq i \leq l \end{cases}$$

Letting $z_i$ denote their images in $A$, note that the new variables $Z_i$ satisfy that $\mathcal{M} = (z_1, z_2, \ldots, z_l)$, $\mathcal{M}/\mathcal{M}^2$ is generated by $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$ and that $\bar{z}_i = 0, n+1 \leq i \leq l$ which means that $z_{n+1}, \ldots, z_l \in \mathcal{M}^2$. Note that this implies that $z_1, \ldots, z_n$ are algebraically independent.

Now we use a step which is used in proving Noether normalisation for arbitrary fields. This allows us to choose our variables without any change in the properties we have already established above for them. Remember that so far $A$ is generated by $Z_i$ as an $\mathbb{R}$-algebra.

Consider the element $z_l$ over $\mathbb{R}[z_1, z_2, \ldots, z_{l-1}]$. Since $z_1, z_2, \ldots, z_n$ are algebraically independent variables in $A$ and $A$ has dimension $n$, $z_l$ must satisfy a polynomial $g$ over $\mathbb{R}[z_1, z_2, \ldots, z_{l-1}]$, i.e. there exists a polynomial $g \in \mathbb{R}[Z_1, Z_2, \ldots, Z_l]$ with $deg_Z(g) > 0$ such that $g(z_1, z_2, \ldots, z_l) = 0$ in $A$.

Now, by a standard lemma ([14, Theorem 13.2]), we can choose sufficiently large $e$ such that if $Z'_i = Z_i - Z_i^e$, then $g$ viewed as a polynomial in the variables $Z'_1, Z'_2, \ldots, Z'_{l-1}, Z_l$ becomes monic in $Z_l$. Hence, $Z_l$ satisfies a monic polynomial over $\mathbb{R}[Z'_1, Z'_2, \ldots, Z'_{l-1}, g]$. Hence, taking images in $A$, we get that $z_l$ satisfies a monic polynomial over $A_{l-1} = \mathbb{R}[z'_1, z'_2, \ldots, z'_{l-1}]$. Let $Z'_l = Z_l$ and $A_l = A$. Then, $A$ is still generated by $z'_1$ and under these new variables, $\mathcal{M}$ still corresponds to the origin and modulo $\mathcal{M}^2$ is still generated by $z'_1, z'_2, \ldots, z'_n$. But that means they are still algebraically independent and we can repeat the above process for $z'_{l-1}$ w.r.t the previous variables.

Thus, repeating this process $l - n$ times, we end up with variables $X_i$ which preserve the properties w.r.t. the maximal ideal. Putting $A_i = \mathbb{R}[x_1, x_2, \ldots, x_i]$, we have that $A_i$ is integral over $A_{i-1}, n + 1 \leq i \leq l$. Hence, $A = A_l$ is integral over $A_0 = \mathbb{R}[x_1, x_2, \ldots, x_n]$ where we also know that $x_i$ are actually algebraically independent.

Thus, we have variables $X_1, X_2, \ldots, X_l$ such that $\mathcal{M} = (x_1, x_2, \ldots, x_l)$, $\mathcal{M}/\mathcal{M}^2$ is generated by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ and $x_{n+1}, \ldots, x_l \in \mathcal{M}^2$ and further $A$ is
integral and hence a finitely generated module over $\mathbb{R}[x_1, \ldots, x_n]$.

To finish the proof, note that $L_M$ is trivial, $MA_M = (x_1, x_2, \ldots, x_n)A_M$ by Nakayama’s lemma and $\Omega_{A_M/R[x_1, \ldots, x_n]} = 0$ since $\Omega_{A_M/R}$ is generated by $dx_1, \ldots, dx_n$. Hence, we can find $f_1, f_2, f_3 \notin M$ such that $L_{f_i}$ is trivial, $M_{f_2} = (x_1, x_2, \ldots, x_n)_{f_2}$ and $\Omega_{A_{f_3}/R[x_1, \ldots, x_n]} = 0$. But then choosing $f = f_1 f_2 f_3 \notin M$, it is clear that the equations above can be simultaneously achieved. Hence, we get the result. \hfill \square

Remark B.1.2. Notice that in the proof, we also got that we could take the point corresponding to $M$ as the origin and that $(x_{l+1}, x_{l+2}, \ldots, x_n) \subseteq M^2$.

We now mention what the above lemma implies topologically. First of all, we can assume that our maximal ideal corresponds to the origin in $\mathbb{R}^l$. Next, we can assume that there exists a function $f$ not passing through the origin and such that in the (semialgebraic) open set $f \neq 0$ of $X(\mathbb{R})$, the line bundle is trivial. Further, the tangent space at every point in this open set is generated by $dX_1, dX_2, \ldots, dX_n$. To apply the semialgebraic implicit function theorem, let $(x^0, y^0)$ be the origin. Then, the functions $X_{n+1}, \ldots, X_l$ clearly pass through it and further, the matrix w.r.t. these functions is just the identity matrix which is invertible. Hence, one can apply the semialgebraic implicit function theorem, which then gives the existence of a semialgebraic open neighbourhood around the origin (which we can choose to be in the set $f \neq 0$) which maps semialgebraically and homeomorphically under the projection map on the first $n$ coordinates. We can actually say a bit more than this, since our functions are actually just coordinate functions which are polynomials, they are also Nash, and hence we can say that there is a Nash isomorphism between a neighbourhood of the origin contained in $f \neq 0$ and its image under the projection map.
Bibliography


