Projective modules over smooth, affine varieties over Archimedean real closed fields

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Abstract

Let $X = \text{Spec}(A)$ be a smooth, affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers. Let P be a projective A-module of rank n such that its n^{th} Chern class $C_n(P) \in \text{CH}_0(X)$ is zero. In this set-up, Bhatwadekar-Das-Mandal showed (amongst many other results) that $P \simeq A \oplus Q$ in the case that either n is odd or the topological space $X(\mathbb{R})$ of real points of X does not have a compact, connected component. In this paper, we prove that similar results hold for smooth, affine varieties over an Archimedean real closed field $\mathbf{R}$.

Key words : Projective modules, Euler Class Groups, real closed fields, semi-algebraically connected semi-algebraic components.

Mathematics Subject Classifications : 13C10, 14P10

1 Introduction

Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field k and let P be a projective A-module of rank n. It is well-known that in general P may not split off a free summand of rank one. Hence, it is of interest to find sufficient conditions for this to happen. When k is an algebraically closed field, a result of Murthy [9, Theorem 3.8] says that if the top Chern class $C_n(P)$ in $\mathrm{CH}_0(X)$ is zero, then P splits off a free summand of rank one (i.e. $P \simeq A \oplus Q$). Note that over any base field, the vanishing of the top Chern class is a necessary condition for P to split off a free summand of rank one. However, the example of the tangent bundle of an even dimensional sphere shows that this condition is not sufficient. Therefore, it is natural to ask : *under what further conditions* $C_n(P) = 0 \overset{?}{\Rightarrow} P \simeq A \oplus Q$. In the case $k = \mathbb{R}$, this question was intially investigated in [2] and brought to a satisfactory conclusion in [1], e.g. it has been shown (amongst many other results) in [1, Theorem 4.30] that when n is odd, then $C_n(P) = 0$ implies that $P \simeq A \oplus Q$. Moreover it is also shown that in the case n is even, $\wedge^n(P) \not\simeq K_A$, then $C_n(P) = 0$ is a sufficient condition for P to have a free summand of rank one, where K_A denotes the canonical module of A over $\mathbb{R}$. In this paper we extend these results to the case when the base field k is an Archimedean real closed field. More precisely, we prove :

Theorem 1.1. *Let* $\mathbf{R}$ *be an Archimedean real closed field. Let* $X = \mathrm{Spec}(A)$ *be a smooth affine variety of dimension* $n \geq 2$ *over* $\mathbf{R}$*. Let* $X(\mathbf{R})$ *denote the* $\mathbf{R}$*-rational points of the variety. Let* K *denote the canonical module* $\wedge^n(\Omega^*_{A/\mathbf{R}})$*. Let* P *be a projective* A*-module of rank* n *and let* $\wedge^n(P) = L$*. Assume that* $C_n(P) = 0$ *in* $\mathrm{CH}_0(X)$*. Then* $P \simeq A \oplus Q$ *in the following cases:*

1. $X(\mathbf{R})$ *has no closed and bounded semi-algebraically connected component.*

2. *For every closed and bounded semi-algebraically connected component* W *of* $X(\mathbf{R})$*,* $L_W \not\simeq K_W$ *where* K_W *and* L_W *denote restriction of (induced) line bundles on* $X(\mathbf{R})$ *to* W*.*

3. n *is odd.*

Moreover, if n *is even and* L *is a rank* 1 *projective* A*-module such that there exists a closed and bounded semi-algebraically connected component* W *of* $X(\mathbf{R})$ *with the property that* $L_W \simeq K_W$*, then there exists a projective* A*-module* P *of rank* n *such that* $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ *(hence* $C_n(P) = 0$*) but* P *does not have a free summand of rank* 1*.*

Note that when the base field is $\mathbb{R}$, the semi-algebraically connected semi-algebraic components are the connected components of $X(\mathbb{R})$ in the Euclidean topology.

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2 Preliminaries

The first part can be looked upon as a quick reference guide to the theory of real closed fields and the topological notions related to them. More details can be found in [4].

Definition 2.1. A field $\mathbf{R}$ is said to be real if it can be ordered in a way such that addition and multiplication are compatible with the ordering. An equivalent definition is that $\sum_{i=1}^{n} a_i^2 = 0 \Rightarrow a_i = 0 \forall i$. A real closed field is a real field which has no algebraic extensions which are real, equivalently attaching a root of -1 makes it algebraically closed.

Such fields come with a natural topology based on intervals like in the case of $\mathbb{R}$. However, under this topology, the field itself is not connected (except in the case of $\mathbb{R}$).

Definition 2.2. A subset V of $\mathbf{R}^n$ is called a basic semi-algebraic set if V is of the form

$$\{x \in \mathbf{R}^n | f_i(x) = 0, g_j(x) > 0, 1 \le i \le r, 1 \le j \le s\},$$

where $f_i(x), g_j(x) \in \mathbf{R}[X_1, X_2, \ldots, X_n]$. A subset W of $\mathbf{R}^n$ is called a semi-algebraic set if W is a finite union of basic semi-algebraic sets.

A semi-algebraic subset W of $\mathbf{R}^n$ is semi-algebraically connected if for every pair of disjoint, closed, semi-algebraic subsets F_1 and F_2 of W, satisfying $F_1 \cup F_2 = W$, either $F_1 = W$ or $F_2 = W$.

Now we quote a result, the proof of which can be found in [4, Theorem 2.4.4].

Theorem 2.3. *Every semi-algebraic subset W of $\mathbf{R}^l$ is the disjoint union of a finite number of semi-algebraically connected semi-algebraic subsets $W_1, W_2,$*

$\ldots, W_s$ which are closed in W. The $W_1, W_2, \ldots, W_s$ are called the ***semi-algebraically connected semi-algebraic components*** *of W.*

Remark 2.4. When the field is $\mathbb{R}$, the semi-algebraically connected semi-algebraic components are same as the connected components by [4, Theorem 2.4.5].

Let $\mathbf{R} \hookrightarrow \mathbf{R}'$ be real closed fields. Let $X = \mathrm{Spec}(A)$ be a smooth affine variety over $\mathbf{R}$ and let $\mathrm{X}(\mathbf{R})$ denote the set of $\mathbf{R}$-rational points of X. Let $A' = A \otimes_{\mathbf{R}} \mathbf{R}'$ and let $X' = \mathrm{Spec}(A')$ be the corresponding (smooth) affine variety over $\mathbf{R}'$. Note that, fixing a closed embedding of $X(\mathbf{R})$ in $\mathbf{R}^l$ (for suitable l), we can regard the topological space $X(\mathbf{R})$ as a subspace of $X(\mathbf{R}')$. Let $W_1', W_2', \ldots, W_s'$ be the semi-algebraically connected semi-algebraic components of $X(\mathbf{R}')$. Let $W_i = W_i' \cap X(\mathbf{R})$. Then, $W_1, W_2, \ldots, W_s$ are precisely the semi-algebraically connected semi-algebraic components of $X(\mathbf{R})$ (for a proof of a more general result see [4, Proposition 5.3.6]). Note that W_i' is closed and bounded if and only if W_i closed and bounded.

Now we state the Artin-Lang homomorphism theorem ([4, Thm. 4.1.2]).

Theorem 2.5. *Let A be a finite type $\mathbf{R}$-algebra. If there exists an $\mathbf{R}$-algebra homomorphism $\phi : A \to \mathbf{R}'$ into a real closed extension $\mathbf{R}'$ of $\mathbf{R}$, then there exists an $\mathbf{R}$-algebra homomorphism $\psi : A \to \mathbf{R}$.*

In particular, if A is an $\mathbf{R}$-subalgebra of $\mathbf{R}'$, then we get a retraction from A to $\mathbf{R}$.

To make the paper self-contained, we define the Euler Class Group. Once again, more details can be obtained in either [1] or [3].

Definition 2.6. Definition of $\mathrm{E}(A, L)$ and $\mathrm{E}_0(A, L)$
Let A be a ring of dimension $n \geq 2$ and let L be a projective A-module of rank 1. Write $F = L \oplus A^{n-1}$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from F/JF to J/J^2 are said to be related if there exists $\sigma \in SL_{A/J}(F/JF)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from F/JF to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local L-orientation* of J. By abuse of notation, we shall identify an equivalence class

$[\alpha]$ with α. A local L-orientation α is called a *global L-orientation* if $\alpha : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : F \twoheadrightarrow J$.

Let G be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements and $\omega_{\mathcal{N}}$ is a local L-orientation of $\mathcal{N}$. Now let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J be a local L-orientation of J. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. We associate to the pair (J, ω_J), the element $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G where $\omega_{\mathcal{N}_i}$ is the local orientation of $\mathcal{N}_i$ induced by ω_J. By abuse of notation, we denote $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ by (J, ω_J).

Let H be the subgroup of G generated by the set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global L-orientation of J.The Euler class group of A with respect to L is $\mathrm{E}(A, L) \stackrel{\text{def}}{=} G/H$. We write $E(A)$ for $E(A, A)$.

Further, let G_0 be the free abelian group on the set $(\mathcal{N})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. Let (J) denote the element $\sum_i (\mathcal{N}_i)$ of G_0. Let H_0 be the subgroup of G_0 generated by elements of the type (J), where J is an ideal of height n such that there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$. Then, $\mathrm{E}_0(A, L) \stackrel{\text{def}}{=} G_0/H_0$. From the definitions of $\mathrm{E}(A, L)$ and $\mathrm{E}_0(A, L)$, it is clear that there is a canonical surjection from $\mathrm{E}(A, L) \twoheadrightarrow \mathrm{E}_0(A, L)$.

Now let P be a projective A-module of rank n such that $L \simeq \wedge^n(P)$ and let $\chi : L \stackrel{\simeq}{\to} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height n. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : L/JL \oplus (A/J)^{n-1} \simeq P/JP$ be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the local L-orientation of J given by $\overline{\varphi} \circ \overline{\gamma} : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$. Let $e(P, \chi)$ be the image in $E(A, L)$ of the element (J, ω_J) of G. The assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A, L)$ is well defined. The *Euler class* of (P, χ) is defined to be $e(P, \chi)$.

We note that for an affine ring over a real closed field $\mathbf{R}$, a maximal ideal $\mathfrak{m}$ of A is called real if $A/\mathfrak{m} \simeq \mathbf{R}$. Now we state a few results for later use.

We begin with a lemma which can be easily deduced from the Eisenbud-Evans theorem as stated in [10, pg. 1420].

Lemma 2.7. *Let R be a Noetherian ring and let K be a proper ideal of height d such that K/K^2 is generated by d elements. Then we can choose $g_1, \ldots, g_d$ which are generators of K/K^2 and $ht.(g_1, \ldots, g_{d-1}) = d - 1$.*

Proof. Choose $f_1, \ldots, f_d$ which generate K/K^2. Then, $(f_1, \ldots, f_d) + K^2 = K$. Hence, there exists $e \in K^2$ such that $(f_1, \ldots, f_d, e) = K$. Since $ht.(K) = ht.(f_1, \ldots, f_d, e) = d$, by the Eisenbud-Evans theorem, there exists $c_1, c_2, \ldots, c_d$ such that $ht.(f_1 + c_1 e, \ldots, f_d + c_d e) = d$. Let $f_i' = f_i + c_i e$. Again, using the Eisenbud-Evans theorem, there is a $b_1, b_2, \ldots, b_{d-1}$ such that if $g_i = f_i' + b_i f_d'$, then $ht.(g_1, \ldots, g_{d-1}) = d - 1$. Let $g_d = f_d'$. Then $(g_1, \ldots, g_d, e) = K$. □

As an application, we deduce the following :

Lemma 2.8. *Let A be a smooth, affine domain over a real closed field $\mathbf{R}$ and let I be a proper ideal of height $n \geq 2$ such that I/I^2 is generated by n elements and I is not contained in any real maximal ideal of A. Then there exists $\{a_1, a_2, \ldots, a_{n-1}\} \subset I$ such that*

1. *$ht.(a_1, a_2, \ldots, a_{n-1}) = n - 1$.*
2. *$(a_1, a_2, \ldots, a_{n-1})$ is not contained in any real maximal ideal of A.*
3. *$I/(a_1, a_2, \ldots, a_{n-1})$ is an invertible ideal in $A/(a_1, a_2, \ldots, a_{n-1})$*

Proof. Let $b_1, b_2, \ldots, b_n$ be elements of I which generate I/I^2. Then $(b_1, b_2, \ldots, b_n) + I^2 = I$. Since I does not belong to any real maximal ideal, we can find an element $b \in I$ which does not belong to any real maximal ideal. By [2, Lemma 4.1] (which is true for any real closed field), there exists $c \in A$ such that the element $a_1 = b_1 + cb^2$ does not belong to any real maximal ideal of A. Let $R = A/(a_1)$ and let $K = I/(a_1)$. Then $ht.(K) = n - 1$ and as $(a_1, b_2, \ldots, b_n) + I^2 = I$, K/K^2 is generated by $n - 1$ elements. Therefore, by (2.7), there exist $a_2, \ldots, a_n \in I$ such that $ht.(\overline{a_2}, \ldots, \overline{a_{n-1}}) = n-2$ and $(\overline{a_2}, \ldots, \overline{a_{n-1}}, \overline{a_n}) + K^2 = K$. Hence, $ht.(a_1, a_2, \ldots, a_{n-1}) = n - 1$ and $(a_1, a_2, \ldots, a_{n-1}, a_n) + I^2 = I$. Further, $(a_1, a_2, \ldots, a_{n-1})$ is not contained in any real maximal ideal of A.

Now note that $A/(a_1, a_2, \ldots, a_{n-1})$ is Cohen-Macaulay (since it is a quotient of a regular ring by a complete intersection), $I/(a_1, a_2, \ldots, a_{n-1})$ has height 1 and is locally generated by a_n. Hence, $I/(a_1, a_2, \ldots, a_{n-1})$ is an invertible ideal. This completes the proof of the lemma. $\square$

The proof of the following lemma is exactly same as [1, Lemma 4.28] when $\mathbf{R} = \mathbb{R}$.

Lemma 2.9. *Let B be an affine ring of dimension 1 over a real closed field $\mathbf{R}$ such that B does not have any real maximal ideal. Let L_1 be a projective B- module of rank 1 and let J be an invertible ideal of B. Then for any positive integer r there exists an invertible ideal J_1 of B such that $J + J_1 = B$ and $L_1 \simeq J \cap {J_1}^r$.*

The following result is proved in [3, Lemma 5.4].

Lemma 2.10. *Let A be a Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n and ω_J be a local L-orientation of J. Let $\overline{a} \in A/J$ be a unit. Then $(J, \omega_J) = (J, \overline{a^2}\omega_J)$ in $E(A, L)$.*

We conclude this section by stating a result of Bhatwadekar-Raja Sridharan which is crucial for the results to follow ([3, Corollary 4.4])

Theorem 2.11. *Let A be a ring of dimension $n \geq 2$ containing the field $\mathbb{Q}$ of rationals. Let L be a projective A-module of rank 1 and P be a projective A-module of rank n with $L \simeq \wedge^n(P)$. Let $\chi : L \xrightarrow{\simeq} \wedge^n P$ be an isomorphism. Let $J \subset A$ be an ideal of height n and ω_J be a local L-orientation of J. Then,*

1. *Suppose that the image of (J, ω_J) is zero in $E(A, L)$. Then there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$ such that ω_J is induced by α (in other words, ω_J is a global L-orientation).*

2. *$P \simeq Q \oplus A$ for some projective A-module Q of rank $n-1$ if and only if $e(P, \chi) = 0$ in $E(A, L)$.*

3 Structure of the Euler Class Group

We give the set-up which will be used subsequently :

Let $\mathbf{R}$ be a real closed field. Let $X = \text{Spec}(A)$ be an affine variety of dimension $n \geq 2$ over $\mathbf{R}$. Let $X(\mathbf{R})$ denote the set of the real maximal ideals of A (the set of $\mathbf{R}$-rational points of X). Let $\mathbf{R}(X)$ denote the localization of A with respect to the multiplicatively closed subset of A consisting of all elements which do not belong to any real maximal ideal. Note that since $\mathbf{R}$ is real closed, $X(\mathbf{R})$ can be thought of as the set of maximal ideals of $\mathbf{R}(X)$. Moreover, by ([4, Theorem 4.4.5]), $\mathbf{R}(X) = S_A^{-1}(A)$, where $S_A = \{1 + \sum_{i=1}^{p} f_i^2 | f_1, f_2, \ldots, f_p \in A\}$.

For a finitely generated projective A-module of rank 1, L, we denote $L_X = L \otimes_A \mathbf{R}(X)$. Throughout this paper, we assume that $X(\mathbf{R})$ is not empty.

Now let X be smooth. The group of zero cycles of X modulo rational equivalence will be denoted by $\text{CH}_0(X)$. Recall that if I is an ideal in A of height n such that I/I^2 is generated by n elements, then $[I]$ denotes the cycle associated to A/I in $\text{CH}_0(X)$ and (I) denotes an element of $\text{E}_0(A, L)$ associated to I.

From the definition of $\text{CH}_0(X)$, it is clear that there exists a natural surjection $\text{E}_0(A, L) \twoheadrightarrow \text{CH}_0(X)$. Then, the induced map $\Theta_L : \text{E}(A, L) \twoheadrightarrow \text{CH}_0(X)$ has the property that if P is a projective A-module of rank n with an isomorphism $\chi : \wedge^n(P) \simeq L$, then $\Theta_L(e(P, \chi)) = C_n(P)$, where $C_n(P)$ denotes the n^{th} Chern class of P (which is an element of $\text{CH}_0(X)$). Therefore, in view of (2.11), in order to conclude that $P \simeq A \oplus Q$ given $C_n(P) = 0$, it is enough to prove that $e(P, \chi) = 0$ for some χ, and hence computation of the Euler class group $\text{E}(A, L)$ is crucial.

Further, there exists a natural surjection $\Gamma_L : \text{E}(A, L) \twoheadrightarrow \text{E}(\mathbf{R}(X), L_X)$. Note that by [1, Lemma 2.7], it can be shown that every element of $\ker \Gamma_L$ is of the form (I, ω_I) where I is an ideal of height n not contained in real maximal ideal and ω_I is a local L-orientation. Hence, we denote $\ker \Gamma_L$ by $\text{E}^{\bar{\mathbf{R}}}(L)$ where $\bar{\mathbf{R}}$ denotes the algebraic closure of $\mathbf{R}$. We first show that $\text{E}^{\bar{\mathbf{R}}}(L)$ is a torsion-free and divisible group, and hence $\text{E}(A, L) \simeq \text{E}(\mathbf{R}(X), L_X) \oplus \text{E}^{\bar{\mathbf{R}}}(L)$.

Let $A_{\bar{\mathbf{R}}} = A \otimes_{\mathbf{R}} \bar{\mathbf{R}}$ and $Y = \text{Spec}(A_{\bar{\mathbf{R}}})$. Then $A_{\bar{\mathbf{R}}}$ is a smooth affine $\bar{\mathbf{R}}$-algebra of dimension $n \geq 2$ and hence $\text{CH}_0(Y)$ is a divisible group. Let $\pi : Y \rightarrow X$

be the canonical map. Note that π is a finite morphism and it induces group homomorphisms $\pi^* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ and $\pi_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0\, 0(X)$ such that the composition $\pi_*\pi^*$ is multiplication by 2. Let $G = \pi_*(\mathrm{CH}_0(Y))$. Then, G is torsion-free and divisible (see [1, Lemma 4.25] for a proof). Therefore, it suffices to show that $\mathrm{E}^{\bar{\mathbf{R}}}(L) \simeq G$. In view of the description of the elements of $\mathrm{E}^{\bar{\mathbf{R}}}(L)$, it is easy to see that the image of $\mathrm{E}^{\bar{\mathbf{R}}}(L)$ under the map $\Theta_L : \mathrm{E}(A, L) \to \mathrm{CH}_0(X)$ is contained in G and therefore it suffices to prove the following proposition.

Proposition 3.1. $(\Theta_L)|_{(\mathrm{E}^{\bar{\mathbf{R}}}(L))}$ *is an isomorphism with image G.*

Remark 3.2. This proposition has been proved in [1] in the case $\mathbf{R} = \mathbb{R}$. The proof in [1] depends on two key results namely (i)$\mathrm{E}_0(A, L) \simeq \mathrm{CH}_0(X)$ and (ii)the topological result that every topological line bundle is 2-torsion on a real manifold. Though our proof is on similar lines, we circumvent these results. More precisely, we do not use (i), and instead of (ii), we prove that any rank 1 projective module over $\mathbf{R}(X)$ is 2-torsion.

Proof. We first prove that the map $\mathrm{E}^{\bar{\mathbf{R}}}(L) \to G$ is injective. Let $x = (I, \omega_I) \in \mathrm{E}^{\bar{\mathbf{R}}}(L)$ such that $\Theta_L((I, \omega_I)) = 0$ where I is an ideal of A of height n contained in only non-real maximal ideals. First note that since I is contained in only non-real maximal ideals, every unit of A/I is a square. Therefore, in view of (2.10), in order to prove that $(I, \omega_I) = 0$ in $E(A, L)$ it is enough to show that there is a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow I$.

Since I is not contained in any real maximal ideals and I/I^2 is generated by n elements, using (2.8) we obtain $\{a_1, \ldots, a_{n-1}\} \subset I$ such that $(a_1, a_2, \ldots, a_{n-1})$ is not contained in any real maximal ideal of A, $I/(a_1, a_2, \ldots, a_{n-1})$ is an invertible ideal in $A/(a_1, a_2, \ldots, a_{n-1})$ and $ht.(a_1, a_2, \ldots, a_{n-1}) = n - 1$.

Let $B = A/(a_1, \cdots a_{n-1})$ and $J = I/(a_1, \cdots a_{n-1})$. Then B is an affine ring of dimension 1 over $\mathbf{R}$ which does not have any real maximal ideal and J is an invertible ideal of B. Hence, letting $r = 2(n-1)!$, lemma 2.9 gives an invertible ideal J_1 of B such that $J + J_1 = B$ and $J \cap {J_1}^r$ is a surjective image of $L \otimes_A B$. Let I_1 be an ideal of A such that $(a_1, \cdots a_{n-1}) \subset I_1$ and $I_1/(a_1, \cdots a_{n-1}) = J_1$. Let ${I_1}^{(r)} = (a_1, \cdots a_{n-1}) + {I_1}^r$. From construction it is clear that there exists a surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow I \cap {I_1}^{(r)}$. Thus, the class corresponding to $I \cap {I_1}^{(r)}$ is

0 in $\mathrm{E}(A, L)$. Hence, if we can prove that there is a surjection $L \oplus A^{n-1} \twoheadrightarrow {I_1}^{(r)}$, then we are done.

Now, we know that $\Theta_L((I, \omega_I)) = [I] = 0$ and note that the image of $(I \cap {I_1}^{(r)}, *)$ under Θ_L is also 0. Hence, $[{I_1}^{(r)}] = 0$. But, as I_1 is contained only in non-real maximal ideals, $[I_1]$ also lies in G, and $[{I_1}^{(r)}] = r[I_1]$ and so $r[I_1] = 0$. Since G is torsion-free, $[I_1] = 0$.

By [6, Theorem 4.1], we obtain that $(n-1)!(I_1) = ({I_1}^{((n-1)!)}) = 0$ in $\mathrm{E}_0(A)$. Now by [1, Lemma 3.6] and [1, Proposition 3.7], we have a natural map $\mathrm{E}_0(A, L) \to \mathrm{E}(A, L)$ so that $({I_1}^{((n-1)!)}) \mapsto ({I_1}^{(2(n-1)!)}, *)$ where $*$ denotes an orientation of ${I_1}^{(2(n-1)!)}$. Recall that $r = 2(n-1)!$. Hence, $({I_1}^{(r)}, *) = 0$ in $\mathrm{E}(A, L)$, which means $\exists L \oplus A^{n-1} \twoheadrightarrow {I_1}^{(r)}$. Hence we have proved injectivity.

Surjectivity follows by imitating proofs of [1, Lemma 4.2] and [1, Lemma 4.26] once we show that if $Z = \mathrm{Spec}(B)$ is a smooth, affine domain of dimension 1 over $\mathbf{R}$ and $\mathfrak{n}$ is a maximal ideal of $\mathbf{R}(Z)$, then $\mathfrak{n}^2$ is a principal ideal. The next result shows that indeed such is the case. This completes the proof of the isomorphism. □

Proposition 3.3. *Let* $\mathbf{R}$ *be a real closed field and let* $Z = \mathrm{Spec}(B)$ *be an affine algebra over* $\mathbf{R}$. *Let* E *be a finitely generated projective module of rank* 1 *over* $\mathbf{R}(Z)$. *Then,* $E \otimes_{\mathbf{R}(Z)} E \xrightarrow{\sim} \mathbf{R}(Z)$.

Thus, the group of rank one projective $\mathbf{R}(Z)$*-modules is* 2*-torsion.*

Proof. We first note that if R is a ring and $\mathfrak{m}$ is a maximal ideal of R such that $R/\mathfrak{m}$ is a real field, then $\sum_{i=1}^n a_i^2 \in \mathfrak{m}$ implies that $\sum_{i=1}^n \bar{a}_i{}^2 = 0$ in $R/\mathfrak{m} \Rightarrow \bar{a}_i = 0\ \forall i \Rightarrow a_i \in \mathfrak{m}\ \forall i$. Thus, $\sum_{i=1}^n a_i^2 \in \mathfrak{m} \Longrightarrow a_i \in \mathfrak{m}\ \forall i$.

Let $(e_1, e_2, \ldots, e_n)$ be a set of generators for E. We claim that $e = \sum_{i=1}^n e_i \otimes e_i$ generates $E \otimes_{\mathbf{R}(Z)} E$. To check this, it is enough to check it in every localization at a maximal ideal. All maximal ideals are real. Let $\mathfrak{m}$ be one such. Then $E_{\mathfrak{m}}$ is a free module of rank 1, hence generated by e_i for some $i, 1 \le i \le n$. Without loss of generality, we assume that it is generated by e_1. Then, $E_{\mathfrak{m}} \otimes_{\mathbf{R}(Z)_{\mathfrak{m}}} E_{\mathfrak{m}}$ is generated by $e_1 \otimes e_1$. Further, $e_i = a_i e_1$, $a_i \in \mathbf{R}(Z)_{\mathfrak{m}}$ and $a_1 = 1$. Then, $e = \sum_{i=1}^n e_i \otimes e_i = (\sum_{i=1}^n a_i^2)(e_1 \otimes e_1)$. Since $1 = a_1$ and the residue field of the local ring is $\mathbf{R}$, $\sum_{i=1}^n a_i^2$ is a unit in $\mathbf{R}(Z)_{\mathfrak{m}}$. Thus, $E_{\mathfrak{m}} \otimes_{\mathbf{R}(Z)_{\mathfrak{m}}} E_{\mathfrak{m}}$ is generated by $e = \sum_{i=1}^n e_i \otimes e_i$. Thus, we see that $E \otimes_{\mathbf{R}(Z)} E \xrightarrow{\sim} \mathbf{R}(Z)$. □

As a consequence of the above proposition, we have the following generalisation of [1, Lemma 4.2] which can be proved along similar lines.

Corollary 3.4. *Let $\mathbf{R}$ be a real closed field and let $X = \operatorname{Spec}(A)$ be an affine variety of dimension $n \geq 2$. Let L_X be a $\mathbf{R}(X)$-projective module of* rank 1 *and I be an ideal of $\mathbf{R}(X)$ of height n such that I/I^2 is generated by n elements. Let ω_I be a local L_X-orientation. Then, $(I, \omega_I) + (I, -\omega_I) = 0$ in $\mathrm{E}(\mathbf{R}(X), L_X)$. Hence, the canonical surjection $\mathrm{E}(\mathbf{R}(X), L_X) \twoheadrightarrow \mathrm{E}_0(\mathbf{R}(X), L_X)$ factors through $\mathrm{E}(\mathbf{R}(X), L_X)/2\,\mathrm{E}(\mathbf{R}(X), L_X)$.*

In view of (3.1), to understand the structure of $\mathrm{E}(A, L)$, we need to investigate the structure of $\mathrm{E}(\mathbf{R}(X), L_X)$. When the base field is $\mathbb{R}$, the following structure theorem has been proved in [1, Theorem 4.21] in terms of compact, connected components of $X(\mathbb{R})$.

Theorem 3.5. *Let $Y = \operatorname{Spec}(B)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ and let $K = \wedge^n(\Omega^*_{B/\mathbb{R}})$ be the canonical module of B. Let E be a projective B-module of rank 1. Let $W_1, \cdots, W_r, W_{r+1}, \cdots, W_t$ be the compact connected components of $Y(\mathbb{R})$ in the Euclidean topology. Let K_{W_i} and E_{W_i} denote restriction of (induced) line bundles on $Y(\mathbb{R})$ to W_i. Assume that $E_{W_i} \simeq K_{W_i}$ for $1 \leq i \leq r$ and $E_{W_i} \not\simeq K_{W_i}$ for $r+1 \leq i \leq t$. Then,*

$$\mathrm{E}(\mathbf{R}(Y), E \otimes_B \mathbf{R}(Y)) = G_1 \oplus \cdots \oplus G_r \oplus G_{r+1} \oplus \cdots \oplus G_t,$$

where $G_i = \mathbb{Z}$ for $1 \leq i \leq r$ and $G_i = \mathbb{Z}/(2)$ for $r+1 \leq i \leq t$. Moreover, given any $\mathfrak{m} \in W_i$ and an E-orientation $\omega_{\mathfrak{m}}$, G_i is generated by $(\mathfrak{m}, \omega_{\mathfrak{m}})$ for $1 \leq i \leq t$.

Using this structure theorem, we now deduce a similar structure theorem when the base field $\mathbf{R}$ is an Archimedean real closed field. Note that every Archimedean field is uniquely order isomorphic to a subfield of $\mathbb{R}$ and hence we may assume it is actually contained in $\mathbb{R}$.

We begin with some generalities. Let $k \hookrightarrow k'$ be a field extension. Let $X = \operatorname{Spec}(A)$ be an affine variety of dimension $n \geq 2$ over k and let $A' = A \otimes_k k'$, $X' = \operatorname{Spec}(A')$. Let Υ be the indexing set for all finite type algebras C_α over k contained in k'. Then $\varinjlim_{\{\alpha \in \Upsilon\}} C_\alpha = k'$ and hence, $A' = \varinjlim_{\{\alpha \in \Upsilon\}} A_\alpha$ where $A_\alpha = A \otimes_k C_\alpha$.

Let L be a projective A-module of rank 1 and $L' = L \otimes_A A'$. Since A' is a ring of dimension n which is faithfully flat over A, it is easy to see that there exist canonical maps $\mathrm{E}(A, L) \to \mathrm{E}(A', L')$ and $\mathrm{E}_0(A, L) \to \mathrm{E}_0(A', L')$. Moreover, if k and k' are real closed fields, then we can extend these maps canonically to maps $\mathrm{E}(k(X), L_X) \to \mathrm{E}(k'(X'), L'_{X'})$ and $\mathrm{E}_0(k(X), L_X) \to \mathrm{E}_0(k'(X'), L'_{X'})$.

Now, we state a lemma, the proof of which is very easy and hence, we only outline the proof.

Lemma 3.6. *Let $\{A_\alpha | \alpha \in \Upsilon\}$ be a direct system of rings. Let $A = \varinjlim_{\{\alpha \in \Upsilon\}} A_\alpha$.*

1. *Let M be a finitely presented A-module. Then, there exists $\alpha \in \Upsilon$ and a finitely presented A_α-module M_α such that $M \cong M_\alpha \otimes_{A_\alpha} A$.*

2. *Let $f : M \to N$ be a homomorphism between two finitely presented A-modules. Then, there exists $\alpha \in \Upsilon$, finitely presented A_α-modules M_α, N_α and an A_α-homomorphism $f_\alpha : M_\alpha \to N_\alpha$ such that $M \cong M_\alpha \otimes_{A_\alpha} A$, $N \cong N_\alpha \otimes_{A_\alpha} A$ and*

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ \downarrow \wr & & \downarrow \wr \\ M_\alpha \otimes_{A_\alpha} A & \xrightarrow{f_\alpha} & N_\alpha \otimes_{A_\alpha} A \end{array}$$

In particular, if f is an epimorphism, then we can choose f_α to be an epimorphism.

Proof. Since M is a finitely presented module, there is an exact sequence $A^n \xrightarrow{h} A^m \to M \to 0$. Then the map h is given by an $m \times n$ matrix H with coefficients in A. Since $A = \varinjlim_{\{\alpha \in \Upsilon\}} A_\alpha$, $\exists \alpha \in \Upsilon$ such that the entries of H come from A_α, i.e. there is a matrix H_α with entries in A_α which map to the corresponding entries of the matrix H. Let $h_\alpha : A_\alpha^n \to A_\alpha^m$ be the corresponding map. Let M_α be the cokernel of the map given by $A_\alpha^n \xrightarrow{h_\alpha} A_\alpha^m$. Then, M_α is finitely presented and $M \cong M_\alpha \otimes_{A_\alpha} A$. This proves the first part. For the second part, let g be the composite map $A^m \to M \xrightarrow{f} N$. Choose M_β and N_β as in the first part. Then, one can define a map $g_\beta : A_\beta^m \to N_\beta$ so that $g_\beta \otimes_{A_\beta} A = g$. Then, the composite map $g_\beta \circ h_\beta : A_\beta^n \to N_\beta$ is such that $g_\beta \circ h_\beta \otimes_{A_\beta} A = g \circ h = 0$.

Hence, by the property of the direct limit, and since n is finite, there exists $\alpha \succ \beta$ such that $g_\alpha \circ h_\alpha = (g_\beta \otimes_{A_\beta} A_\alpha) \circ (h_\beta \otimes_{A_\beta} A_\alpha) = g_\beta \circ h_\beta \otimes_{A_\beta} A_\alpha = 0$. Since $M_\alpha = M_\beta \otimes_{A_\beta} A_\alpha$ is the cokernel of $h_\alpha : A_\alpha^n \to A_\alpha^m$, there is an induced map $f_\alpha : M_\alpha \to N_\alpha$. Clearly, this map satisfies the required property. Further, if f was an epimorphism, then so is g and so one can choose g_β to be an epimorphism which will make g_α and hence f_α an epimorphism. Hence, proved. □

Now we work in the following set-up : $\mathbf{R}$ is an Archimedean real closed field and $X = \mathrm{Spec}(A)$ is a smooth, affine variety of dimension $n \geq 2$ over $\mathbf{R}$. Let $B = A \otimes_{\mathbf{R}} \mathbb{R}$ and let $Y = \mathrm{Spec}(B)$. Let L be a rank 1 projective A-module and let $L' = L \otimes_A B$. Note that $L'_Y = L_X \otimes_{\mathbf{R}(X)} \mathbb{R}(Y)$.

Proposition 3.7. *The canonical maps* $\mathrm{E}(\mathbf{R}(X), L_X) \to \mathrm{E}(\mathbb{R}(Y), L'_Y)$ *and* $\mathrm{E}_0(\mathbf{R}(X), L_X) \to \mathrm{E}_0(\mathbb{R}(Y), L'_Y)$ *are isomorphisms. In particular,* $\mathrm{E}_0(\mathbf{R}(X), L_X)$ *is a vector space of* rank t *over the field* $\mathbb{Z}/(2)$ *where* t *is the number of closed and bounded semi-algebraically connected semi-algebraic components of* $X(\mathbf{R})$.

Proof. We first prove the isomorphism for $\mathrm{E}(\mathbf{R}(X), L_X) \to \mathrm{E}(\mathbb{R}(Y), L'_Y)$.
(i) Injectivity : Let J be an ideal of height n in $\mathbf{R}(X)$ such that J/J^2 is generated by n elements and let ω_J be a surjective map from $F = L_X \oplus \mathbf{R}(X)^{n-1} \twoheadrightarrow J/J^2$ (a local L-orientation). Let $I = J\mathbb{R}(Y)$. Now suppose there is a surjective map $\theta_I : F \otimes_{\mathbf{R}(X)} \mathbb{R}(Y) \twoheadrightarrow I$ which is a lift of $\omega_J \otimes_{\mathbf{R}(X)} \mathbb{R}(Y)$. Then, by lemma 3.6, there exists an affine $\mathbf{R}$-subalgebra C of $\mathbb{R}$ such that if $T = \{1 + \sum_{i=1}^{p} f_i^2 | f_1, f_2, \dots, f_p \in A \otimes_{\mathbf{R}} C\}$, and $D = T^{-1}(A \otimes_{\mathbf{R}} C)$, then there is a surjection $\theta : F \otimes_{\mathbf{R}(X)} D \twoheadrightarrow JD$, which is a lift of $\omega_J \otimes_{\mathbf{R}(X)} D$.

Since $\mathbf{R} \hookrightarrow C \hookrightarrow \mathbb{R}$, by (2.5), $\mathbf{R}$ is a retract of C and hence, A is a retract of $A \otimes_{\mathbf{R}} C$. Let $\beta : A \otimes_{\mathbf{R}} C \to A$ be an A-algebra homomorphism which gives rise to this retract. Let $S_A = \{1 + \sum_{i=1}^{p} f_i^2 | f_1, f_2, \dots, f_p \in A\}$. Since $S_A \subset T$ and $\beta(T) = S_A$, $\mathbf{R}(X) \subset D$ and β induces a retraction $\sigma : D \twoheadrightarrow \mathbf{R}(X)$. In view of this, it is easy to see that the map $\theta \otimes_D \mathbf{R}(X)$ is indeed a lift of ω_J. Thus, the map $\mathrm{E}(\mathbf{R}(X), L_X) \to \mathrm{E}(\mathbb{R}(Y), L'_Y)$ is injective.

(ii) Surjectivity : Let $W_1, \cdots, W_r, W_{r+1}, \cdots, W_t$ be the compact connected components of $Y(\mathbb{R})$ in the Euclidean topology. Then by (3.5), we get that

$E(\mathbb{R}(Y), L'_Y) = \oplus_{i=1}^t G_i$. Further, given any $\mathfrak{m} \in W_i$ and an L'_Y-orientation $\omega_{\mathfrak{m}}$, G_i is generated by $(\mathfrak{m}, \omega_{\mathfrak{m}})$ for $1 \leq i \leq t$. Note that by (2.4), if W is a connected component of $Y(\mathbb{R})$, then $W \cap X(\mathbf{R}) \neq \emptyset$. Hence, there exists a maximal ideal $\mathfrak{m}_i$ of $\mathbf{R}(X)$, such that $\mathfrak{m}_i \in W_i \cap X(\mathbf{R})$. Now we choose any L_X-orientation ω_i of $\mathfrak{m}_i$. Then,(m_i, ω_i) maps to a generator of G_i. Hence, $E(\mathbf{R}(X), L_X) \to E(\mathbb{R}(Y), L'_Y)$ is surjective.

Using [2, Theorem 4.10] and (3.5), it is easy to see that the induced map $E(\mathbb{R}(Y), L'_Y)/2\,E(\mathbb{R}(Y), L'_Y) \to E_0(\mathbb{R}(Y), L'_Y)$ is an isomorphism.

Therefore, the composite map $E(\mathbf{R}(X), L_X)/2\,E(\mathbf{R}(X), L_X) \to E_0(\mathbb{R}(Y), L'_Y)$ is an isomorphism. But by (3.4), we know that this map is actually the composite map $E(\mathbf{R}(X), L_X)/2\,E(\mathbf{R}(X), L_X) \twoheadrightarrow E_0(\mathbf{R}(X), L_X) \to E_0(\mathbb{R}(Y), L'_Y)$. Hence, $E_0(\mathbf{R}(X), L_X) \to E_0(\mathbb{R}(Y), L'_Y)$ is an isomorphism. Hence, by [2, Theorem 4.10] and (2.4), we see that $E_0(\mathbf{R}(X), L_X)$ is a vector space of rank t over the field $\mathbb{Z}/(2)$ where t is the number of closed and bounded semi-algebraically connected semi-algebraic components of $X(\mathbf{R})$. □

Remark 3.8. Let $\mathbf{R} \hookrightarrow \mathbf{R}'$ be an extension of real closed fields and let $X = \mathrm{Spec}(A)$ be an affine variety over $\mathbf{R}$ and let L be a rank 1 projective A-module. Let $A' = A \otimes_{\mathbf{R}} \mathbf{R}'$, $X' = \mathrm{Spec}(A')$ and $L' = L \otimes_A A'$. Then the above proof shows that $E(A, L) \to E(A', L')$ and $E(\mathbf{R}(X), L_X) \to E(\mathbf{R}'(X'), L'_{X'})$ are both injective. As a consequence, we have the following result.

Corollary 3.9. *Let* $\mathbf{R}$ *be a real closed field and let*

$$B = \mathbf{R}[X_0, X_1, \ldots, X_n]/(\sum_{i=0}^{n} X_i^2 - 1)$$

denote the coordinate ring of the sphere $S^n(\mathbf{R})$. *Assume that* n *is even. Let* Q *be a projective* B*-module corresponding to the tangent bundle of* $S^n(\mathbf{R})$. *Then* Q *does not split off a free summand of* rank 1.

Proof. Let K denote the algebraic closure of $\mathbb{Q}$ in $\mathbf{R}$. It is easy to see that K is isomorphic to $\bar{\mathbb{Q}} \cap \mathbb{R}$ where $\bar{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $A = K[X_0, X_1, \ldots, X_n]/(\sum_{i=0}^n X_i^2 - 1)$. Let P be the kernel of the map $A^{n+1} \twoheadrightarrow A$ sending e_i to $\bar{X}_i$. Then $\wedge^n(P) \simeq A$. Let χ be a generator of $\wedge^n(P)$.

Note that $B = A \otimes_K \mathbf{R}$ and $Q = P \otimes_A B$. Let $\chi' = \chi \otimes_A B$. To prove the result, it is enough to show that $e(Q, \chi') \neq 0$ in $\mathrm{E}(B)$. But since $e(Q, \chi')$ is the image of $e(P, \chi)$ under the *injective* map $E(A) \hookrightarrow E(B)$, it is enough to show that $e(P, \chi) \neq 0$ in $\mathrm{E}(A)$. But since K is an Archimedean field, by a result of Kong [8, Theorem 6.1], $e(P, \chi) \neq 0$. $\square$

We return back to the set-up we were in, namely that $\mathbf{R}$ is an Archimedean real closed field, and $X = \mathrm{Spec}(A)$ is a smooth, affine variety over $\mathbf{R}$. Let $W_1', W_2', \ldots, W_t'$ be the compact connected components of $Y(\mathbb{R})$. Then, in view of (3.7) and (3.5), we know that $E(\mathbf{R}(X), L_X) = \oplus_{i=1}^t G_i$ where $G_i = \mathbb{Z}$ or $G_i = \mathbb{Z}/(2)$. Let $W_i = W_i' \cap X(\mathbf{R})$. By (2.4), $W_1, W_2, \ldots, W_t$ are precisely the closed and bounded semi-algebraically connected semi-algebraic components of $X(\mathbf{R})$. Let $x \in W_i$ and let $\mathfrak{m}$ be the real maximal ideal of $\mathbf{R}(X)$ corresponding to x. Let $\omega_{\mathfrak{m}}$ be an L_X-orientation of $\mathfrak{m}$. Let G_i be the cyclic subgroup of $\mathrm{E}(\mathbf{R}(X), L_X)$ generated by $(\mathfrak{m}, \omega_{\mathfrak{m}})$. We denote by $\mathcal{L}_i$ the semi-algebraic line bundle on W_i induced by L_X and by $\mathcal{L}_i'$, the semi-algebraic line bundle on W_i' induced by L_Y'. Now we prove some results which give us a criterion to determine whether $G_i = \mathbb{Z}$ or $G_i = \mathbb{Z}/(2)$ in terms of the semi-algebraic line bundle $\mathcal{L}_i$. We start with a lemma in this direction.

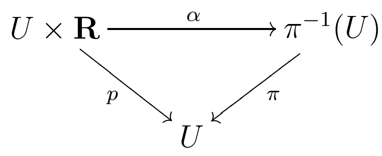
Lemma 3.10. *Let V be a semi-algebraically connected semi-algebraic set and let $\pi : \mathcal{E} \to V$ be a semi-algebraic line bundle. Then, $\mathcal{E}^* = \mathcal{E} \setminus \{zero\ section\}$ has 2 semi-algebraically connected semi-algebraic components if and only if $\mathcal{E}$ is semi-algebraically trivial.*

Proof. If $\mathcal{E}$ is semi-algebraically trivial, it is clear that $\mathcal{E}^* = \mathcal{E} \setminus \{zero\ section\}$ has two semi-algebraically connected semi-algebraic components. We now prove the other part.

In view of [4, Lemma 12.7.3], it is easy to see that every semi-algebraic vector bundle comes equipped with a continuous, semi-algebraic map $\mathcal{E} \oplus \mathcal{E} \xrightarrow{b} \mathbf{R}$ (called a Riemannian metric) such that restricted to each fibre, it is an inner product. Let θ be the composite map $\mathcal{E} \xrightarrow{\Delta} \mathcal{E} \oplus \mathcal{E} \xrightarrow{b} \mathbf{R}$ induced through the diagonal. Then, $\mathcal{E}^* = \theta^{-1}(0, \infty)$. Let $\mathcal{E}^* = \mathcal{U}_1 \sqcup \mathcal{U}_2, \mathcal{U}_1 \neq \phi, \mathcal{U}_2 \neq \phi$, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are semi-algebraic open subsets.

Step 1 : We shall prove that for every fibre the intersection with $\mathcal{U}_1$ and $\mathcal{U}_2$ is non-empty. Let $V_1 = \{x \in V : \mathcal{U}_1 \cap \mathcal{E}_x \neq \phi, \mathcal{U}_2 \cap \mathcal{E}_x \neq \phi\} = \pi(\mathcal{U}_1) \cap \pi(\mathcal{U}_2)$
$V_2 = \{x \in V : \mathcal{U}_1 \cap \mathcal{E}_x = \phi\} = \{x \in V : \mathcal{E}^*_x \subseteq \mathcal{U}_2\} = \pi(\mathcal{U}_2) \cap (\pi(\mathcal{U}_1))^c$ and
$V_3 = \{x \in V : \mathcal{U}_2 \cap \mathcal{E}_x = \phi\} = \{x \in X : \mathcal{E}^*_x \subseteq \mathcal{U}_1\} = \pi(\mathcal{U}_1) \cap (\pi(\mathcal{U}_2))^c$.
Since a line bundle is locally trivial, all the three sets are clearly open in V. Since π is semi-algebraic, these sets are also semi-algebraic. Again, they are clearly disjoint and span X. Since X is semi-algebraically connected, $V = V_i$ for some i. If $V = V_2$, then $\mathcal{E}^*_x \subseteq \mathcal{U}_2$ $\forall x \in X \Rightarrow \mathcal{E}^* = \mathcal{U}_2 \Rightarrow \mathcal{U}_1 = \phi$ which is a contradiction. Hence, $V \neq V_2$. Similarly, $V \neq V_3$. Hence, $V = V_1$. Therefore, given $x \in V$, $\mathcal{U}_1 \cap \mathcal{E}_x \neq \phi$ and $\mathcal{U}_2 \cap \mathcal{E}_x \neq \phi$. But $\mathcal{E}^*_x$ has exactly two semi-algebraically connected components. Therefore, they are exactly $\mathcal{U}_1 \cap \mathcal{E}_x$ and $\mathcal{U}_2 \cap \mathcal{E}_x$.
Step 2 : We will now define a section. Let $x \in V$ and restrcict θ to $\mathcal{E}_x$. This is the square of a norm on the 1-dimensional vector space over $\mathbf{R}$. Therefore, there exist two elements of $\mathcal{E}_x$ such that their evaluation under θ is 1. Further, if one of them is t, then the other is $-t$. From step 1, exactly one of these elements lies in $\mathcal{U}_1$. In other words, $\theta^{-1}(1) \cap \mathcal{E}_x \cap \mathcal{U}_1$ is a singleton set. Define the map $\psi : V \to \mathcal{E}^*$ sending x to the element of $\theta^{-1}(1) \cap \mathcal{E}_x \cap \mathcal{U}_1$.
Step 3 : We will now show that ψ is continuous and semi-algebraic. We first note that there is a finite, open, semi-algebraically connected, semi-algebraic covering of our base space V such that the bundle is trivial on them. Then, it is enough to prove that for each such open set U, $\psi|_U$ is continuous and semi-algebraic.

Let U be one of the open sets in the cover and let α be a local trivialization of $\mathcal{E}$ over U, i.e.



Now, note that $\pi^{-1}(U) \cap \mathcal{E}^*$ has exactly two semi-algebraically connected components, $\pi^{-1}(U) \cap \mathcal{U}_1$ and $\pi^{-1}(U) \cap \mathcal{U}_2$ which correspond to $U \times \mathbf{R}^+$ and $U \times \mathbf{R}^-$ through α. W.l.g. we can always assume that $U \times \mathbf{R}^+ \xrightarrow{\sim} \pi^{-1}(U) \cap \mathcal{U}_1$ (else we choose $-\alpha$). Then, it is enough to prove the composite

$$U \xrightarrow{\psi} \pi^{-1}(U) \cap \mathcal{U}_1 \overset{\alpha^{-1}}{\xrightarrow{\sim}} U \times \mathbf{R}^+$$

is continuous and semi-algebraic. Let $w \in U$. Consider $\theta(\alpha(w,1)) = r(w) > 0$. Then r is a continuous, semi-algebraic function on U. Therefore, we get

$$\frac{1}{r(w)}\theta(\alpha(w,1)) = 1 \Rightarrow \theta(\frac{1}{\sqrt{r(w)}}\alpha(w,1)) = 1 \Rightarrow \theta(\alpha(w,\frac{1}{\sqrt{r(w)}})) = 1.$$

$$\Rightarrow \alpha(w,\frac{1}{\sqrt{r(w)}}) \in \theta^{-1}(1) \cap \mathcal{E}_x \cap \mathcal{U}_1$$

$$\Rightarrow \psi(w) = \alpha(w,\frac{1}{\sqrt{r(w)}})$$

$$\Rightarrow \alpha^{-1}(\psi(w)) = (w,\frac{1}{\sqrt{r(w)}})$$

Hence, we are reduced to showing that $w \mapsto (w, \frac{1}{\sqrt{r(w)}})$ is continuous and semi-algebraic. But this is true since r was a continuous, non-zero, semi-algebraic function. Hence, we are done. □

Let $K_A = \wedge^n(\Omega^*_{A/\mathbf{R}})$ and $K_B = \wedge^n(\Omega^*_{B/\mathbb{R}})$. We denote by $\mathcal{K}_i$ the semi-algebraic line bundle on W_i induced by K_A and by $\mathcal{K}'_i$, the semi-algebraic line bundle on W'_i induced by K_B.

Lemma 3.11. *$\mathcal{L}_i \simeq \mathcal{K}_i$ if and only if $\mathcal{L}'_i \simeq \mathcal{K}'_i$.*

Proof. Let $E = L \otimes_A K_A$ and $E' = L' \otimes_B K_B$. Let $\mathcal{E}$ be the line bundle on $X(\mathbf{R})$ corresponding to E and $\mathcal{E}'$ be the line bundle on $Y(\mathbb{R})$ corresponding to E'. Let $\mathcal{E}_i$ be the restriction of $\mathcal{E}$ to W_i and $\mathcal{E}'_i$ be the restriction of $\mathcal{E}'$ to W'_i. Then $\mathcal{E}_i = \mathcal{L}_i \otimes \mathcal{K}_i$ and $\mathcal{E}'_i = \mathcal{L}'_i \otimes \mathcal{K}'_i$. In view of (3.3), it is enough to prove that $\mathcal{E}_i$ is semi-algebraically trivial if and only if $\mathcal{E}'_i$ is trivial. We proceed to show this.

Note first that in $\mathbb{R}$, semi-algebraic triviality of a semi-algebraic line bundle is same as topological triviality (remark 2.4). Without loss of generality, let $\mathcal{E}_i$ be semi-algebraically trivial if and only if $1 \leq i \leq q$. Then, by lemma 3.10,

${\mathcal{E}_i}^*$ has only one semi-algebraically connected semi-algebraic component when $sq+1 \leq i \leq s$ and 2 components otherwise. Thus, $\mathcal{E}^*$ has exactly $2q+(s-q)$ semi-algebraically connected components. But then using remark 2.4, we know that $\mathcal{E}'^*$ has exactly the same no. of connected components. But $\mathcal{E}_i$ is semi-algebraically trivial implies that $\mathcal{E}'_i$ is trivial. Hence, $\mathcal{E}'_1, \mathcal{E}'_2, \ldots, \mathcal{E}'_q$ are all trivial and so each of ${\mathcal{E}'_1}^*, {\mathcal{E}'_2}^*, \ldots, {\mathcal{E}'_q}^*$ has 2 connected components. Thus, ${\mathcal{E}'_i}^*$ must have a single connected component for $q+1 \leq i \leq s$ and so $\mathcal{E}'_i$ must be non-trivial. Hence, we see that $\mathcal{E}'_i$ is trivial whenever $\mathcal{E}_i$ is trivial and $\mathcal{E}'_i$ is non-trivial whenever $\mathcal{E}_i$ is non-trivial. Therefore, the lemma stands proved. $\square$

Now, using (3.7) and (3.11), we clearly obtain a structure theorem for the Euler class group $\mathrm{E}(\mathbf{R}(X), L_X)$, which is very similar to (3.5), viz.

Theorem 3.12. *Let* $\mathbf{R}$ *be an Archimedean real closed field. Let* $X = \mathrm{Spec}(A)$ *be a smooth affine variety of dimension* $n \geq 2$ *over* $\mathbf{R}$ *and let* $K = \wedge^n(\Omega^*_{A/\mathbf{R}})$ *be the canonical module of* A*. Let* L *be a projective* A*-module of rank* 1 *and* $L_X = L \otimes_A \mathbf{R}(X)$*. Let* $W_1, \cdots, W_r, W_{r+1}, \cdots, W_t$ *be the closed and bounded semi-algebraically connected components of* $X(\mathbf{R})$ *in the Euclidean topology. Let* K_{W_i} *and* L_{W_i} *denote restriction of (induced) line bundles on* $X(\mathbf{R})$ *to* W_i*. Assume that* $L_{W_i} \simeq K_{W_i}$ *for* $1 \leq i \leq r$ *and* $L_{W_i} \not\simeq K_{W_i}$ *for* $r+1 \leq i \leq t$*. Then,*

$$E(\mathbf{R}(X), L_X) = G_1 \oplus \cdots \oplus G_r \oplus G_{r+1} \oplus \cdots \oplus G_t,$$

where $G_i = \mathbb{Z}$ *for* $1 \leq i \leq r$ *and* $G_i = \mathbb{Z}/(2)$ *for* $r+1 \leq i \leq t$*.*

Moreover, for $1 \leq i \leq t$*, if* $x \in W_i$ *and* $\mathfrak{m}$ *is a maximal ideal of* $\mathbf{R}(X)$ *corresponding to* x*, then* G_i *is generated by* $(\mathfrak{m}, \omega_{\mathfrak{m}})$ *for any local* L_X*-orientation* $\omega_{\mathfrak{m}}$ *of* $\mathfrak{m}$*.*

Remark 3.13. A special case of the above theorem, namely $L = K$ has already been proved by Ian Robertson [11, Theorem 12.6].

4 Main Theorem

In this section, we prove the main theorem (Theorem 1.1). We first recall our set-up and some notations. Let $\mathbf{R}$ denote an Archimedean real closed field and let $\bar{\mathbf{R}}$ denote the algebraic closure of $\mathbf{R}$ in $\mathbb{C}$. Let $X = \operatorname{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbf{R}$. Let $A_{\bar{\mathbf{R}}} = A \otimes_{\mathbf{R}} \bar{\mathbf{R}}$ and $\bar{X} = \operatorname{Spec}(A_{\bar{\mathbf{R}}})$. Let $\pi : \bar{X} \to X$ be the canonical map and $\pi_* : \mathrm{CH}_0(\bar{X}) \to \mathrm{CH}_0(X)$. Let $G = \pi_*(\mathrm{CH}_0(\bar{X}))$. Let L be a rank 1 projective A-module.

Let $\Gamma_L : \mathrm{E}(A, L) \twoheadrightarrow \mathrm{E}(\mathbf{R}(X), L_X)$ and $\mathrm{E}^{\bar{\mathbf{R}}}(L) = \ker(\Gamma_L)$. Let Θ_L denote the map $\mathrm{E}(A, L) \twoheadrightarrow \mathrm{CH}_0(X)$. Recall that $\Theta_L(\mathrm{E}^{\bar{\mathbf{R}}}(L)) \subseteq G$ and in fact by (3.1), $\Theta_L|_{\mathrm{E}^{\bar{\mathbf{R}}}(L)} : \mathrm{E}^{\bar{\mathbf{R}}}(L) \to G$ is an isomorphism. For the sake of convenience, we denote $\Theta_L|_{\mathrm{E}^{\bar{\mathbf{R}}}(L)}$ by Ψ_L. Keeping this notation in mind, we have the following commutative diagram :

$$\begin{array}{ccccccccc}
 & & & & \ker(\Theta_L) & \xrightarrow{\sim} & \ker(\Phi_L) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{E}^{\bar{\mathbf{R}}}(L) & \longrightarrow & \mathrm{E}(A, L) & \xrightarrow{\Gamma_L} & \mathrm{E}(\mathbf{R}(X), L_X) & \longrightarrow & 0 \\
 & & {\scriptstyle \Psi_L}\downarrow\wr & & {\scriptstyle \Theta_L}\downarrow & & {\scriptstyle \Phi_L}\downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & \mathrm{CH}_0(X) & \longrightarrow & \mathrm{CH}_0(X)/G & \longrightarrow & 0
\end{array}$$

Lemma 4.1. $\ker(\Phi_L)$ *(*$\simeq \ker(\Theta_L)$*) is a free abelian group.*

Proof. By a result of Colliot-Thélène and Scheiderer [5, Theorem 1.3(d)], the group $\mathrm{CH}_0(X)/G$ is a vector space of dimension t over the field $\mathbb{Z}/(2)$ where t is the number of closed and bounded semi-algebraically connected semi-algebraic components of $X(\mathbf{R})$. Therefore, $2\,\mathrm{E}(\mathbf{R}(X), L_X)$ is contained in $\ker(\Phi_L)$. But by (3.7), $\mathrm{E}(\mathbf{R}(X), L_X)/2\,\mathrm{E}(\mathbf{R}(X), L_X)$ is itself a vector space of dimension t over $\mathbb{Z}/(2)$. Therefore, $\ker(\Phi_L) = 2\,\mathrm{E}(\mathbf{R}(X), L_X)$ which is a free abelian group. □

Lemma 4.2. *The canonical map* $\mathrm{E}_0(A, L) \twoheadrightarrow \mathrm{CH}_0(X)$ *is an isomorphism.*

Proof. Similar to the above diagram, we have another commutative diagram as follows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {\mathrm{E}_0}^{\bar{\mathbf{R}}}(L) & \longrightarrow & \mathrm{E}_0(A, L) & \longrightarrow & \mathrm{E}_0(\mathbf{R}(X), L_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & \mathrm{CH}_0(X) & \longrightarrow & \mathrm{CH}_0(X)/G & \longrightarrow & 0
\end{array}$$

Note that the map $\mathrm{E}^{\bar{\mathbf{R}}}(L) \to G$ factors through $\mathrm{E_0}^{\bar{\mathbf{R}}}(L)$ and the map $\mathrm{E}^{\bar{\mathbf{R}}}(L) \to \mathrm{E_0}^{\bar{\mathbf{R}}}(L)$ is surjective. Hence, the induced map $\mathrm{E}^{\bar{\mathbf{R}}}(L) \twoheadrightarrow \mathrm{E_0}^{\bar{\mathbf{R}}}(L)$ is an isomorphism and therefore, $\mathrm{E}^{\bar{\mathbf{R}}}(L) \to G$ is also an isomorphism. Now, (3.7) and the result of Colliot-Thélène and Scheiderer [5, Theorem 1.3(d)] imply that $\mathrm{E}_0(\mathbf{R}(X), L_X)$ and $\mathrm{CH}_0(X)/G$ are vector spaces over $\mathbb{Z}/(2)$ of the same dimension. Hence, the map $\mathrm{E}_0(\mathbf{R}(X), L_X) \twoheadrightarrow \mathrm{CH}_0(X)/G$ is an isomorphism. Then, using the 5-lemma in the above diagram, we see that $\mathrm{E}_0(A, L) \twoheadrightarrow \mathrm{CH}_0(X)$ is an isomorphism. □

Theorem 4.3. *Let* $\mathbf{R}$ *be an Archimedean real closed field. Let* $X = \mathrm{Spec}(A)$ *be a smooth affine variety of dimension* $n \geq 2$ *over* $\mathbf{R}$*. Let* $X(\mathbf{R})$ *denote the* $\mathbf{R}$*-rational points of the variety. Let* K *denote the canonical module* $\wedge^n(\Omega^*_{A/\mathbf{R}})$*. Let* P *be a projective* A*-module of rank* n *and let* $\wedge^n(P) = L$*. Assume that* $C_n(P) = 0$ *in* $\mathrm{CH}_0(X)$*. Then* $P \simeq A \oplus Q$ *in the following cases:*

1. $X(\mathbf{R})$ *has no closed and bounded semi-algebraically connected component.*
2. *For every closed and bounded semi-algebraically connected component* W *of* $X(\mathbf{R})$*,* $L_W \not\simeq K_W$ *where* K_W *and* L_W *denote restriction of (induced) line bundles on* $X(\mathbf{R})$ *to* W*.*
3. n *is odd.*

Moreover, if n *is even and* L *is a rank* 1 *projective* A*-module such that there exists a closed and bounded semi-algebraically connected component* W *of* $X(\mathbf{R})$ *with the property that* $L_W \simeq K_W$*, then there exists a projective* A*-module* P *of rank* n *such that* $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ *(hence* $C_n(P) = 0$*) but* P *does not have a free summand of rank* 1*.*

Though the proof of the above is same as the proof of [1, Theorem 4.30], we repeat the arguments to make the paper self-contained.

Proof. Let P be a projective A-module of rank n with $\wedge^n(P) \simeq L$ and let $\chi : L \to \wedge^n(P)$ be an L-orientation of P. Then, $\Theta_L(e(P,\chi)) = C_n(P)$. In view of (2.11), to prove the theorem, it is enough to prove that $C_n(P) = 0 \Rightarrow e(P,\chi) = 0$ in all three cases.

Proof of 1. and 2.
Note that in this case, by the structure theorem of $\mathrm{E}(\mathbf{R}(X), L_X)$, Φ_L is an isomorphism and hence, Θ_L is an isomorphism.

Proof of 3. When n is odd, there is an automorphism Δ of P with determinant -1. Let $\alpha : P \twoheadrightarrow I$ where I is an ideal of height n. Let ω_I be a local L-orientation of I induced by α. Using this and Δ, we get that $2e(P, \chi) = (I, \omega_I) + (I, -\omega_I)$. Since the canonical map $\mathrm{E}_0(A, L) \twoheadrightarrow \mathrm{CH}_0(X)$ is an isomorphism by (4.2) and $(I) \mapsto C_n(P)$, we have $(I) = 0$ in $\mathrm{E}_0(A, L)$. Therefore, by [1, Proposition 3.7], $2e(P, \chi) = (I, \omega_I) + (I, -\omega_I) = 0$. Since $C_n(P) = 0$, $e(P, \chi) \in \ker(\Theta_L)$, which is a free abelian group by (4.1). Hence, $e(P, \chi) = 0$.

Finally, let n be even, and $L_W \simeq K_W$ for some closed and bounded semi-algebraically connected component W. Then, by our assumption, (3.7) and (4.1), $\ker(\Theta_L) \simeq \ker(\Phi_L) \neq 0$. Since $\mathrm{E}_0(A, L) \to \mathrm{CH}_0(X)$ is an isomorphism, by abuse of notation, we denote the canonical map $\mathrm{E}(A, L) \to \mathrm{E}_0(A, L)$ by Θ_L. Then, as in [2, Lemma 3.3], there exists an ideal J of height n such that J is a surjective image of $L \oplus A^{n-1}$ and a local L-orientation ω_J which is not a global orientation, i.e. $(J, \omega_J) \neq 0$ in $\mathrm{E}(A, L)$. Since n is even, as in [2, Lemma 3.6], we can get a rank n projective module P, which is stably isomorphic to $L \oplus A^{n-1}$ (i.e. $P \oplus A \simeq L \oplus A^{n-1} \oplus A$) and $\chi : L \to \wedge^n(P)$ such that $e(P, \chi) = (I, \omega_I) \neq 0$ in $E(A, L)$. Note that since P is stably isomorphic to $L \oplus A^{n-1}$, $C_n(P) = 0$ but $e(P, \chi) \neq 0$ and hence, by [3, Corollary 4.4], $P \not\simeq Q \oplus A$. This completes the proof.

□

References

[1] S.M.Bhatwadekar, M.K.Das, Satya Mandal, Projective modules over smooth, real, affine varieties, *Invent. Math.* **166** (2006), 151-184.

[2] S. M. Bhatwadekar, Raja Sridharan, Zero cycles and the Euler class groups of smooth real affine varieties, *Invent. Math.* **136** (1999), 287-322.

[3] S. M. Bhatwadekar, Raja Sridharan, The Euler class group of a Noetherian ring, *Compositio Math.* **122**(2000), 183-222.

[4] J. Bochnak, M. Coste, M.-F. Roy, Real algebraic geometry, Ergebnisse der Mathematik, Volume 36, Springer-Verlag, 1998.

[5] J.-L.Colliot-Thélène, C.Scheiderer, Zero cycles and cohomology on real algebraic varieties, *Topology* **35** (1996), 533-559.

[6] Mrinal Kanti Das, Satya Mandal, A Riemann-Roch theorem, *J. Algebra* **301** (2006), 148-164.

[7] D. Eisenbud, E. G. Evans, Generating modules efficiently : Theorems from algebraic K-theory, *J. Algebra* **27** (1973), 278-305.

[8] Maynard Kong, Euler classes of inner product modules, *J. Algebra* **49** (1977), no. 1, 276-303.

[9] M.P.Murthy, Zero cycles and projective modules, Ann Math. **140** (1994), 405-435.

[10] Plumstead, B. : The conjectures of Eisenbud-Evans, Am. J. Math. **105** (1983), 1417-1433

[11] Ian Robertson, The Euler class group of a line bundle on an affine algebraic variety over a real closed field, Ph.D. thesis, University of Chicago, 1999.