Projective Modules over smooth, affine varieties over real closed fields

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Abstract

Let $X = \text{Spec}(A)$ be a smooth, affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers. Let P be a projective A-module of rank n such that its n^{th} Chern class $C_n(P) \in \text{CH}_0(X)$ is zero. In this set-up, Bhatwadekar-Das-Mandal showed (amongst many other results) that $P \simeq A \oplus Q$ in the case that either n is odd or the topological space $X(\mathbb{R})$ of real points of X does not have a compact, connected component. In this paper, we prove that similar results hold for smooth, affine varieties over an arbitrary real closed field $\mathbf{R}$. The proof is algebraic and does not make use of Tarski's principle, nor of the earlier result for $\mathbb{R}$.

Key words : Projective modules, Euler Class Groups, real closed fields, semialgebraically connected semialgebraic components, elementary paths.

Mathematics Subject Classifications : 13C10, 14P10, 14R99

1 Introduction

Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field k of characteristic 0 and let P be a projective A-module of rank n. It is well-known that in general P may not split off a free summand of rank 1 as $C_n(P) = 0$ is a necessary condition, where $C_n(P)$ denotes the n^{th} Chern class of P which is an element of the group $\mathrm{CH}_0(X)$ of zero cycles modulo rational equivalence. A result of Murthy [12, Theorem 3.8] says that when k is an algebraically closed field, $C_n(P) = 0$ is also sufficient. However, if k is not algebraically closed, then $C_n(P) = 0$ is not always a sufficient condition as evidenced by the example of the tangent bundle of the even dimensional real sphere. Hence, it is of interest to know when $C_n(P) = 0$ is a sufficient condition for P to split off a free summand of rank 1. In the case $k = \mathbb{R}$, this question was brought to a satisfactory conclusion in [1]. Subsequently, similar conclusions were proved in [4] when the base field is Archimedean real closed. In this paper we extend these results to the case when the base field k is a real closed field. Before giving a precise statement of our result, we would like to mention that the results in [4] used the existing theorem for $\mathbb{R}$ in [1] and also indirectly made use of Tarski's principle. Further, the proofs in [1] used topological techniques, which crucially used the fact that open intervals in $\mathbb{R}$ are connected (the only real closed field having this property). In fact, open intervals in real closed fields other than $\mathbb{R}$ are totally disconnected. However, our proofs in this paper are completely algebraic (modulo the fact that the statement has a bit of topology), they work uniformly for any real closed field and do not invoke Tarski's principle. Now we state our main result:

Theorem A. *Let* $\mathbf{R}$ *be a real closed field. Let* $X = \mathrm{Spec}(A)$ *be a smooth affine variety of dimension* $n \geq 2$ *over* $\mathbf{R}$*. Let* $X(\mathbf{R})$ *denote the* $\mathbf{R}$*-rational points of the variety. Let* K *denote the module* $\wedge^n(\Omega_{A/\mathbf{R}})$*. Let* P *be a projective* A*-module of rank* n *and let* $\wedge^n(P) = L$*. Assume that* $C_n(P) = 0$ *in* $\mathrm{CH}_0(X)$*. Then* $P \simeq A \oplus Q$ *in the following cases:*

1. $X(\mathbf{R})$ *has no closed and bounded semialgebraically connected component.*

2. *For every closed and bounded semialgebraically connected component* W *of* $X(\mathbf{R})$*,* $L_W \not\simeq K_W$ *where* K_W *and* L_W *denote restriction of (induced) line bundles on* $X(\mathbf{R})$ *to* W*.*

3. n *is odd.*

Moreover, if n is even and L is a rank 1 projective A-module such that there exists a closed and bounded semialgebraically connected component W of $X(\mathbf{R})$ with the property that $L_W \simeq K_W$, then there exists a projective A-module P of rank n such that $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ (hence $C_n(P) = 0$) but P does not have a free summand of rank 1.

Let $\mathbf{R}, X, A, L$ be as in **Theorem A**. Let $\mathbf{R}(A)$ denote the ring of real regular functions, i.e. the ring obtained by inverting all elements which do not belong to any real maximal ideal and let $\mathbf{R}(L) = L \otimes_A \mathbf{R}(A)$.

As in [1], one of the key ingredients in the proof of **Theorem A** is a structure theorem for the Euler class group $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$:a notion due to Nori (see preliminaries for a definition). This structure theorem is in terms of the semialgebraically connected semialgebraic components of the space of real points $X(\mathbf{R})$ (see next section for definitions).

Theorem B. *Let $A, K, L, \mathbf{R}(A)$ be as above. Let $C_i, 1 \leq i \leq t$ be the closed and bounded semialgebraically connected semialgebraic components of $X(\mathbf{R})$. Let L_i and K_i be the restriction of the semialgebraic line bundles corresponding to L and K respectively, to C_i.Let $L_i \simeq K_i$, for $1 \leq i \leq r$ and $L_i \not\simeq K_i$, for $r+1 \leq i \leq t$. Let $x_i \in C_i$ and let $\mathcal{M}_i$ be the corresponding maximal ideal of $\mathbf{R}(A)$. Let ω_i be a local $\mathbf{R}(L)$-orientation of $\mathcal{M}_i$. Then, $\oplus_{i=1}^{r} \mathbb{Z} e_i \bigoplus \oplus_{i=r+1}^{t} (\mathbb{Z}/2) e_i \xrightarrow{\sim} \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ sending $e_i \mapsto (\mathcal{M}_i, \omega_i)$ is an isomorphism.*

We begin this paper with some preliminaries in section 2 which we divide into 3 subsections. The first subsection introduces real closed fields and semialgebraic sets, the second one is about elementary paths and the final one defines the Euler class group and states lemmas which will be required later on in the paper. Since the proof of **Theorem A** (assuming **Theorem B**) is similar to the ones already mentioned in [1] and [4], we sketch the proof in section 3. The proof of the structure theorem **Theorem B** mainly involves the notion of an elementary path as defined in [8, Definition 3.1]. In section 4, we derive a result which shows that the ideal of all functions vanishing at the initial point and endpoint of an elementary path in $X(\mathbf{R})$ is a complete intersection (we can prove something stronger but prove only whatever is necessary for the proof of the structure theorem **Theorem B**). In section 5, we relate the generators of $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ with the total space of a line bundle and prove that points of an elementary path in the bundle are equal in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. Using this we prove the structure theorem **Theorem B** in section 6.

2 Preliminaries

Real Closed Fields and semialgebraic sets

The first part can be looked upon as a quick reference guide to the theory of real closed fields and the topological notions related to them. More details can be found in [5].

Definition 2.1. A field $\mathbf{R}$ is said to be real if it can be ordered in a way such that addition and multiplication are compatible with the ordering. An equivalent definition is that $\sum_{i=1}^{n} a_i^2 = 0 \Rightarrow a_i = 0 \forall i$. A real closed field is a real field which has no algebraic extensions which are real, equivalently attaching a root of -1 makes it algebraically closed.

Such fields come with a natural topology based on intervals like in the case of $\mathbb{R}$. However, under this topology, the field itself is not connected (except in the case of $\mathbb{R}$). We can extend this topology to $\mathbf{R}^l$ (product topology). We call this topology the Euclidean topology. Note that this topology comes from a "metric" taking values in $\mathbf{R}$, namely $d(\underline{x}, \underline{y}) = \sqrt{\sum_{i=1}^{l}(x_i - y_i)^2}$ where $\underline{x} = (x_1, x_2, \ldots, x_l)$ and $\underline{y} = (y_1, y_2, \ldots, y_l)$.

Hence, a subset $V \subset \mathbf{R}^l$ inherits the Euclidean topology and the associated "metric". Thus, one can talk of open, closed and bounded sets in V.

Definition 2.2.

- A subset V of $\mathbf{R}^l$ is called a basic semialgebraic set if V is of the form

  $$\{x \in \mathbf{R}^l | f_i(x) = 0, g_j(x) > 0, 1 \leq i \leq r, 1 \leq j \leq s\},$$

  where $f_i(x), g_j(x) \in \mathbf{R}[X_1, X_2, \ldots, X_l]$. A subset W of $\mathbf{R}^l$ is called a semialgebraic set if W is a finite union of basic semialgebraic sets.

- A semialgebraic subset W of $\mathbf{R}^l$ is semialgebraically connected if for every pair of disjoint, closed, semialgebraic subsets F_1 and F_2 of W satisfying $F_1 \cup F_2 = W$, either $F_1 = W$ or $F_2 = W$.
- A map between two semialgebraic sets $f : A \to B$ is said to be semialgebraic if its graph is a semialgebraic set.
- A semialgebraic path in a semialgebraic set V is the image of a continuous, semialgebraic map $f : [0, 1] \to V$.

Now we quote a result, the proof of which can be found in [5, Theorem 2.4.4].

Theorem 2.3. *Every semialgebraic subset W of $\mathbf{R}^l$ is the disjoint union of a finite number of semialgebraically connected semialgebraic subsets $W_1, W_2, \ldots, W_s$ which are closed in W. The $W_1, W_2, \ldots, W_s$ are called the* ***semi-algebraically connected semialgebraic components*** *of W. By abuse of notation, we shall refer to them simply as components of W.*

Remark 2.4. When the field is $\mathbb{R}$, the semialgebraically connected semialgebraic components are same as the connected components by [5, Theorem 2.4.5].

Two points P and Q of a semialgebraic set W lie in the same component of W if and only if they can be joined by a semialgebraic path in W.

We refer to [5, 12.7.1] for the notion of a semialgebraic vector bundle. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two semialgebraic line bundles, we will denote $\mathcal{E}_1 \simeq \mathcal{E}_2$ to mean that $\mathcal{E}_1$ and $\mathcal{E}_2$ are semialgebraically isomorphic (i.e. the isomorphism between them is also semialgebraic).

With this background, we state a result for later use.

Lemma 2.5. *[4, Lemma 3.10] Let W be a semialgebraically connected semialgebraic set and let $\pi : \mathcal{E} \to W$ be a semialgebraic line bundle. Then, $\mathcal{E}^* = \mathcal{E} \setminus \{zero\ section\}$ has 2 components if and only if $\mathcal{E}$ is a semialgebraically trivial line bundle.*

We set up some notation for the rest of the paper.

A maximal ideal $\mathcal{M}$ of a ring A is called a real maximal ideal if $A/\mathcal{M}$ is a real field. Note that if A is an affine algebra over a real closed field $\mathbf{R}$, then every maximal ideal has residue field either $\mathbf{R}$ or its algebraic closure $\bar{\mathbf{R}}$. The $\mathbf{R}$-rational points are real maximal ideals while we refer to $\bar{\mathbf{R}}$-rational points as non-real maximal ideals (in some places in the literature they are referred to as complex maximal ideals).

Let A be an affine algebra over a real closed field $\mathbf{R}$ and $X = \mathrm{Spec}(A)$. Then we denote by $\mathbf{R}(A)$, the ring obtained by inverting all elements which do not belong to any real maximal ideal. This is same as the localisation $S^{-1}A$ where $S = \{1 + \sum_{i=1}^{n} f_i^2 | f_i \in A\}$. Note that maximal ideals of $\mathbf{R}(A)$ are in one-to-one correspondence with real maximal ideals of A. We denote by $X(\mathbf{R})$ the set of all $\mathbf{R}$-rational points of X. In this paper, very often we do not distinguish between $\mathbf{R}$-rational points of X and the corresponding maximal ideals of $\mathbf{R}(A)$. For a module M over A, we denote the $\mathbf{R}(A)$-module $M \otimes_A \mathbf{R}(A)$ by $\mathbf{R}(M)$. We remark that in literature the ring $\mathbf{R}(A)$ is sometimes referred to as $\mathbf{R}(X)$.

Note that if A is an affine algebra over a real closed field $\mathbf{R}$, then elements of $\mathbf{R}(A)$ act as functions on $X(\mathbf{R})$ taking values in $\mathbf{R}$ (canonically). There is a natural map $sign : \mathbf{R}^* \to \{\pm 1\}$; namely $sign(\lambda) = 1$ if $\lambda > 0$ and $sign(\lambda) = -1$ if $\lambda < 0$. For a function f taking values in $\mathbf{R}^*$, we can then talk about $sign(f(P))$ for any point P in the domain. In this sense, we use $sign(f(P))$ where $P \in X(\mathbf{R})$ and $f \in \mathbf{R}(A)$ does not belong to the maximal ideal of $\mathbf{R}(A)$ corresponding to P.

Elementary Paths

We start by quoting a few results and setting up notations necessary for defining an elementary path. The same proof as that of [4, Propn. 3.3] gives us the next result.

Proposition 2.6. *Let $\mathbf{R}$ be a real closed field and let B be an affine algebra over $\mathbf{R}$. Let E be a projective module of rank 1 over $\mathbf{R}(B)$ generated by $\{e_1, e_2, \ldots, e_n\}$. Then, $\sum_{i=1}^{n} e_i \otimes e_i$ generates $E \otimes_{\mathbf{R}(B)} E \xrightarrow{\sim} \mathbf{R}(B)$. Thus, the group of rank one projective $\mathbf{R}(B)$-modules is 2-torsion.*

Similarly, if B is the coordinate ring of a curve over $\mathbf{R}$, then for any regular maximal ideal $\mathfrak{m}$ of $\mathbf{R}(B)$, if $\mathfrak{m} = (a_1, a_2, \ldots, a_n)$, then $\mathfrak{m}^2 = (\sum_{i=1}^{n} a_i^2)$.

Now, we set up some notations. Let $Z = \mathrm{Spec}(C)$ be a smooth affine curve over a real closed field $\mathbf{R}$. Let $\bar{Z}$ be its smooth projectivisation. Then, we have a natural injection $Z \hookrightarrow \bar{Z}$. Note that Z is an open subset of $\bar{Z}$. Hence, stalks will be isomorphic, and hence the local rings $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{\bar{Z},z}$ will be same for points $z \in Z$. Hence, real points of Z continue to be real points of $\bar{Z}$, i.e. $Z(\mathbf{R}) \hookrightarrow \bar{Z}(\mathbf{R})$.

Since $\mathbf{R}$ is not algebraically closed, all real points of $\bar{Z}$ are actually contained in an affine, open subset of $\bar{Z}$. Let this be $Z' = \mathrm{Spec}(C')$. Then, consider $Z \cap Z' = \tilde{Z}$ which is affine. Let $\tilde{C}$ be the coordinate ring of $\tilde{Z}$. Note that since $Z'(\mathbf{R}) = \bar{Z}(\mathbf{R})$, we have $\tilde{Z}(\mathbf{R}) = Z(\mathbf{R})$.

Let $K(\bar{Z})$ be the function field of $\bar{Z}$. Then, $\mathcal{O}_{\bar{Z},z} \hookrightarrow K(\bar{Z})$ and hence,

$$\mathbf{R}(C') = \cap_{z \in Z'(\mathbf{R})} \mathcal{O}_{\bar{Z},z}, \quad \mathbf{R}(C) = \cap_{z \in Z(\mathbf{R})} \mathcal{O}_{\bar{Z},z} = \cap_{z \in \tilde{Z}(\mathbf{R})} \mathcal{O}_{\bar{Z},z} = \mathbf{R}(\tilde{C}).$$

Thus, $\mathbf{R}(C') \hookrightarrow \mathbf{R}(\tilde{C}) = \mathbf{R}(C)$. Moreover, since $\mathbf{R}(C')$ is a Dedekind domain, birational to $\mathbf{R}(C)$ and $\mathrm{Pic}(\mathbf{R}(C'))$ is two-torsion, $\mathbf{R}(C)$ is a localisation of $\mathbf{R}(C')$. Note that $\bar{Z}$ is a smooth, complete curve and Z' is an affine open subset containing all its real points. With this notation in mind, we quote some theorems from [10].

Theorem 2.7. *[10, Theorem 5.2] $\Omega_{\mathbf{R}(C')/\mathbf{R}}$ is a free module of* rank 1 *over* $\mathbf{R}(C')$.

Fix a generator of $\Omega_{\mathbf{R}(C')/\mathbf{R}}$, say χ, which is regarded as a global "orientation". This continues to be a generator for $\Omega_{\mathbf{R}(\tilde{C})/\mathbf{R}}$. With this notation in mind, we obtain the following :

Theorem 2.8. *[10, 4.5a-6.1-6.2] Given two points P, Q in the same component of $Z'(\mathbf{R})$, there is a function $f_{P,Q} \in \mathbf{R}(C')$ with the following properties :*

1. *$(f_{P,Q}) = \mathfrak{m}_P \cap \mathfrak{m}_Q$ in $\mathbf{R}(C')$.*
2. *if $df_{P,Q} = g\ \chi$, then it has opposite orientations at both points, i.e. $sign(g(P)) = -1, sign(g(Q)) = 1$.*
3. *$f_{P,Q}$ is positive at all points outside the component containing P and Q.*

Remark 2.9. Note that if there are two functions $f_{P,Q}$ and $f'_{P,Q}$ satisfying the above properties, then $f_{P,Q} = uf'_{P,Q}$ where $u \in \mathbf{R}(C')^*$ is such that $u(T) > 0 \forall T \in Z'(\mathbf{R})$ (Artin's Theorem then says that u is a sum of squares of rational functions).

The function $f_{P,Q}$ defines an open interval $]P,Q[=\{T\in Z'(\mathbf{R})|f_{P,Q}(T)<0\}$. Let

$$[P,Q]=\{T\in Z'(\mathbf{R})|f_{P,Q}(T)\leq 0\}=]P,Q[\cup\{P\}\cup\{Q\}$$

be the corresponding closed interval, which is actually the closure of $]P,Q[$ in the Euclidean topology. By definition, these intervals are semialgebraic subsets of $Z'(\mathbf{R})$. They lie in a component of $Z'(\mathbf{R})$ and are semialgebraically connected.

Let $R,S\in]P,Q[$ be distinct points. Then, we can define a total order on $]P,Q[$ by defining $R<S$ if $[P,R]\subseteq[P,S]$. This order naturally extends to $[P,Q]$ by letting $P<R<Q$ for all $R\in]P,Q[$. We refer to [10, Section 6] and [11] for more details.

Note that since $Z(\mathbf{R})\hookrightarrow Z'(\mathbf{R})$ and $\mathbf{R}(C')\hookrightarrow\mathbf{R}(C)$ is a localisation, we have the following facts:

- $f_{P,Q}\in\mathbf{R}(C)$
- $\Omega_{\mathbf{R}(C)/\mathbf{R}}\simeq\Omega_{\mathbf{R}(C')/\mathbf{R}}\otimes\mathbf{R}(C)$ and hence is free
- the components of $Z(\mathbf{R})$ are contained in the components of $Z'(\mathbf{R})$
- if $z\in Z(\mathbf{R})\subseteq Z'(\mathbf{R})$, then the corresponding maximal ideal $\mathfrak{m}_z\subset\mathbf{R}(C')$ satisfies $\mathbf{R}(C')/\mathfrak{m}_z\simeq\mathbf{R}(C)/\mathfrak{m}_z\mathbf{R}(C)$

If $[P,Q]\subset Z(\mathbf{R})$, then it is called a closed interval of $Z(\mathbf{R})$. In that case, $f_{P,Q}$ is positive at all points of $Z(\mathbf{R})$ outside $[P,Q]$, in particular on all the points outside the component containing P and Q.

We now define elementary paths.

Definition 2.10. Let $X=\mathrm{Spec}(A)$ be an affine variety over $\mathbf{R}$. An elementary path in $X(\mathbf{R})$ is a totally ordered subset γ of $X(\mathbf{R})$ which either consists only of one point ("degenerate" elementary path) or has the following two properties :

- The Zariski closure of γ in $\mathrm{Spec}(A)$ is an irreducible curve $\mathrm{Spec}(B)\subset\mathrm{Spec}(A)$.
- If $\Pi:Z=\mathrm{Spec}(C)\twoheadrightarrow\mathrm{Spec}(B)$ denotes the normalisation of $\mathrm{Spec}(B)$, then after a choice of a suitable orientation on $Z(\mathbf{R})$, there exists a bijective and order preserving map from a closed interval $[P,Q]\subset Z(\mathbf{R})$ onto γ.

Remark 2.11. Elementary paths are essentially bijective images of closed intervals in smooth curves onto $X(\mathbf{R})$. We call $\Pi(P)$ the starting point or initial point of γ and $\Pi(Q)$ the endpoint of γ.

Note that every elementary path is a bijective image of $[0,1] \subseteq \mathbf{R}$ ([9, Theorem 10.1]). In particular it implies that intervals as defined above for arbitrary smooth curves are also bijective images of $[0,1]$.

We now quote a theorem that will make it clear why the notion of elementary paths is of importance to us.

Theorem 2.12. *[9, Theorem 10.2] Any semialgebraic path can be broken into finitely many non-degenerate elementary paths $\gamma_i, 1 \leq i \leq r$ such that $\gamma_i \cap \gamma_{i+1} = \{S_i\}$ and S_i is the initial point of γ_{i+1} and the endpoint of γ_i.*

Some algebraic results

To make the paper self-contained, we define the Euler Class Group. We give a definition only in the case where the underlying ring is a smooth affine domain since it will be the definition we use in this paper. More details can be obtained in either [1] or [3].

Definition 2.13. Definition of $\mathrm{E}(A, L)$ and $\mathrm{E}_0(A, L)$
Let A be a smooth affine domain of dimension $n \geq 2$ and let L be a projective A-module of rank 1. Let $\mathcal{M}$ be a maximal ideal of A of height n. Then, $\mathcal{M}/\mathcal{M}^2$ is generated by n elements. An isomorphism $\omega_{\mathcal{M}} : L/\mathcal{M}L \xrightarrow{\sim} \wedge^n(\mathcal{M}/\mathcal{M}^2)$ is called a *local L-orientation* of $\mathcal{M}$. Let G be the free abelian group on the set of pairs $(\mathcal{M}, \omega_{\mathcal{M}})$ where $\mathcal{M}$ is a maximal ideal of height n and $\omega_{\mathcal{M}}$ is a local L-orientation of $\mathcal{M}$.

Let $J = \cap_{i=1}^{k} \mathcal{M}_i$ be an intersection of finitely many maximal ideals of height n. Then, J/J^2 is generated by n elements. An isomorphism $L/JL \xrightarrow{\sim} \wedge^n(J/J^2)$ is called a *local L-orientation* of J. A local L-orientation of J gives rise to local $\mathcal{M}_i$-orientations $\omega_{\mathcal{M}_i}, i = 1, 2, \ldots, k$. Then, we denote the element $\sum_{i=1}^{k}(\mathcal{M}_i, \omega_{\mathcal{M}_i})$ in G as (J, ω_J).

A local L-orientation $\omega : L/JL \twoheadrightarrow \wedge^n(J/J^2)$ is called a *global L-orientation* if there exists a surjection $\theta : L \oplus A^{n-1} \twoheadrightarrow J$, such that ω is the induced isomorphism

$L/JL \overset{\alpha}{\xrightarrow{\sim}} \wedge^n(L/JL \oplus (A/J)^{n-1}) \overset{\wedge^n(\theta)}{\xrightarrow{\sim}} \wedge^n(J/J^2)$ where $\alpha(\bar{e}) = \bar{e} \wedge \bar{e_2} \wedge \ldots \wedge \bar{e_n}$ (and $\{e_2, e_3, \ldots, e_n\}$ is a basis of A^{n-1}).

Let H be the subgroup of G generated by the set of pairs (J, ω_J), where J is a finite intersection of maximal ideals of height n and ω_J is a global L-orientation of J. The Euler class group of A with respect to L is $\mathrm{E}(A, L) \overset{\text{def}}{=} G/H$. We write $E(A)$ for $E(A, A)$.

Further, let G_0 be the free abelian group on the set $(\mathcal{M})$ where $\mathcal{M}$ is a maximal ideal of A. Let $J = \cap_{i=1}^{k} \mathcal{M}_i$ be a finite intersection of maximal ideals. Let (J) denote the element $\sum_i (\mathcal{M}_i)$ of G_0. Let H_0 be the subgroup of G_0 generated by elements of the type (J), where J is a finite intersection of maximal ideals such that there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$. Then, $\mathrm{E}_0(A, L) \overset{\text{def}}{=} G_0/H_0$. From the definitions of $\mathrm{E}(A, L)$ and $\mathrm{E}_0(A, L)$, it is clear that there is a canonical surjection $\mathrm{E}(A, L) \twoheadrightarrow \mathrm{E}_0(A, L)$.

Now let P be a projective A-module of rank n such that $L \simeq \wedge^n(P)$ and let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is a finite intersection of maximal ideals of height n. Therefore we obtain an induced isomorphism $\overline{\varphi} : P/JP \to J/J^2$. Let ω_J be the local L-orientation of J given by $\wedge^n(\overline{\varphi}) \circ \overline{\chi}$. Let $e(P, \chi)$ be the image in $E(A, L)$ of the element (J, ω_J) of G. The assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A, L)$ is well defined. The *Euler class* of (P, χ) is defined to be $e(P, \chi)$.

Remark 2.14. The Euler class group can be defined for any Noetherian, commutative ring A and finitely generated projective module L of rank 1, as done in [3] and in the case of a smooth, affine domain the two definitions coincide.

Note that if ω_0 and ω_1 are two local orientations of a reduced ideal J, then $\omega_0 = \lambda \omega_1$ where $\lambda \in (A/J)^*$.

We state a few theorems for later use. The next couple of lemmas give us some tools to make computations in the Euler class group.

Lemma 2.15. *[3, Lemma 5.4] Let A be a Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n and ω_J be a local L-orientation of J. Let $\overline{a} \in A/J$ be a unit. Then $(J, \omega_J) = (J, \overline{a^2}\omega_J)$ in $E(A, L)$.*

Lemma 2.16. *[1, Lemma 4.3] Let A be a smooth affine domain over $\mathbf{R}$. Let L be a projective A-module of rank 1. Let $\mathcal{M}$ be a maximal ideal of $\mathbf{R}(A)$ and $\omega_{\mathcal{M}}$ be a local L-orientation of $\mathcal{M}$. Then $(\mathcal{M}, \omega_{\mathcal{M}}) + (\mathcal{M}, -\omega_{\mathcal{M}}) = 0$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. As a consequence, $\mathrm{E}_0(\mathbf{R}(A), \mathbf{R}(L))$is a vector space over $\mathbb{Z}/(2)$. Moreover, if $\widetilde{\omega}_{\mathcal{M}}$ is another local L-orientation of $\mathcal{M}$ then either $(\mathcal{M}, \widetilde{\omega}_{\mathcal{M}}) = (\mathcal{M}, \omega_{\mathcal{M}})$ or $(\mathcal{M}, \widetilde{\omega}_{\mathcal{M}}) = (\mathcal{M}, -\omega_{\mathcal{M}})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.*

The next theorem is a crucial theorem which illustrates the purpose behind defining the Euler class group.

Theorem 2.17. *[3, Corollary 4.4] Let A be a ring of dimension $n \geq 2$ containing the field $\mathbb{Q}$ of rationals. Let L be a projective A-module of rank 1 and P be a projective A-module of rank n with $L \simeq \wedge^n(P)$. Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $J \subset A$ be an ideal of height n and ω_J be a local L-orientation of J. Then,*

1. *Suppose that (J, ω_J) is zero in $E(A, L)$. Then there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$ such that ω_J is induced by α (in other words, ω_J is a global L-orientation).*
2. *$P \simeq Q \oplus A$ for some projective A-module Q of rank $n-1$ if and only if $e(P, \chi) = 0$ in $E(A, L)$.*

We now prove a lemma which allows us to analyse the natural map $A/J \twoheadrightarrow A/\sqrt{J}$ when $ht(J) = dim(A)$.

Lemma 2.18. *Let k be a field with characteristic $\neq 2$ and let B be a k-algebra. Let I be a nilpotent ideal of B. Let $g \in B^*$ be such that g has a square root modulo I. Then, $\exists g_1 \in B$ such that $g_1^2 = g$. In particular, if $g \equiv 1 mod\ I$, then g_1 can be so chosen that $g_1 \equiv 1 mod\ I$.*

Proof. Since I is nilpotent, B is complete w.r.t. the I-adic topology (which is actually the discrete topology). We attach a variable Y to B. Let $f(Y) = Y^2 - g$. Let "bar" denote going modulo I. Let $\bar{g} = \bar{h}^2, h \in B$. Since g is a unit, so is h. Then,

$$\bar{f}(Y) = \bar{Y}^2 - \bar{g} = \bar{Y}^2 - \bar{h}^2 = (\bar{Y} - \bar{h})(\bar{Y} + \bar{h})$$

and since characteristic of k is not equal to 2, $\bar{Y}-\bar{h}$ and $\bar{Y}+\bar{h}$ are co-maximal in $B[Y]$ and hence, applying Hensel's lemma, Y^2-g has a solution in B, say g_1 such that $g_1 \equiv \bar{h} mod\ I$. If $g \equiv 1 mod\ I$, then clearly $g_1 \equiv \pm 1 mod\ I$ and hence, we can choose g_1 so that $g_1 \equiv 1 mod\ I$. □

The next lemma allows us to analyse conductor diagrams.

Lemma 2.19. *Suppose $f : B \to B'$ is a monomorphism of rings and let $\mathfrak{c}_{B'/B}$ be the conductor ideal of B' w.r.t. B. Let I be an ideal in B such that $I+\mathfrak{c}_{B'/B}=B$. If $\exists f \in B'$ such that $f \equiv 1 mod \mathfrak{c}_{B'/B}$ and $IB'=fB'$, then $f \in B$ and $I=fB$.*

Proof. Since $f \equiv 1 mod \mathfrak{c}_{B'/B}$, $f-1 \in \mathfrak{c}_{B'/B}$. Let $f-1=x \in \mathfrak{c}_{B'/B} \subseteq B$. Then, $f=x+1 \in B$. Futher, let $y \in I$. Then, $y=fg$, where $g \in B'$. Then, $y=(1+x)g=g+xg$. Now, $x \in \mathfrak{c}_{B'/B} \Rightarrow xg \in \mathfrak{c}_{B'/B} \subseteq B$. Since $y \in I \subseteq B$, we get that $g \in B$. Hence, $y=fg$ implies that $y \in fB$. Hence, $I=fB$. □

Finally, we prove another lemma which will be used later.

Lemma 2.20. *Let $\mathbf{R}$ be a real closed field. Let $Z=\operatorname{Spec}(C)$ be a smooth affine curve over $\mathbf{R}$. If $\mathfrak{m}$ is a non-real maximal ideal of C, then it always satisfies an equation of the form $\mathfrak{m}\prod \mathfrak{m}_j^2(f)=\prod \mathfrak{m}_i^2(g)$ where f and g are sums of squares. In particular, this implies that $(\mathfrak{m}) \in 2\operatorname{Pic}(C)$.*

Proof. Let $\bar{\mathbf{R}}$ be the algebraic closure of $\mathbf{R}$. There is a natural norm map (which is only a multiplicative homomorphism on the units) from $\bar{\mathbf{R}} \twoheadrightarrow \mathbf{R}$ given by $a+bi \mapsto a^2+b^2$ which extends to a natural map $Norm : C \otimes_{\mathbf{R}} \bar{\mathbf{R}} \to C$ given by $f \otimes (a+bi) \mapsto (a^2+b^2)f^2$. Let $\mathfrak{m}$ be a non-real maximal ideal of C. Then there exists a maximal ideal $\mathfrak{n}$ of $C \otimes_{\mathbf{R}} \bar{\mathbf{R}}$ such that $\mathfrak{n} \cap C=\mathfrak{m}$. It is well-known that $\operatorname{Pic}(C \otimes_{\mathbf{R}} \bar{\mathbf{R}})$ is divisible. Hence,

$$(\mathfrak{n})=2\sum_{i=1}^{k_1} s_i(\mathfrak{n}_i)+2\sum_{i=k_1+1}^{k_2} s_i(\mathfrak{n}_i)-2\sum_{i=k_2+1}^{k_3} s_i(\mathfrak{n}_i)-2\sum_{i=k_3+1}^{k_4} s_i(\mathfrak{n}_i)$$

where $s_i>0$ and $\mathfrak{n}_i \cap C=\mathfrak{m}_i$ where

$$\mathfrak{m}_i \text{ is a } \begin{cases} \text{real maximal ideal of } C & 1 \le i \le k_1, k_2+1 \le i \le k_3 \\ \text{non-real maximal ideal of } C & k_1+1 \le i \le k_2 \text{ and } k_3+1 \le i \le k_4 \end{cases}$$

Then, this means that there exist $h_1, h_2 \in C \otimes_{\mathbf{R}} \bar{\mathbf{R}}$ so that

$$(h_1)\mathfrak{n} \prod_{i=k_2+1}^{k_4} \mathfrak{n}_i^{2s_i} = (h_2) \prod_{i=1}^{k_2} \mathfrak{n}_i^{2s_i}.$$

Hence, applying the norm map, we get

$$Norm(h_1)Norm(\mathfrak{n}) \prod_{i=k_2+1}^{k_4} Norm(\mathfrak{n}_i^{2s_i}) = Norm(h_2) \prod_{i=1}^{k_2} Norm(\mathfrak{n}_i^{2s_i})$$

which gives us

$$(Norm(h_1))\mathfrak{m} \prod_{i=k_2+1}^{k_3} \mathfrak{m}_i^{4s_i} \prod_{i=k_3+1}^{k_4} \mathfrak{m}_i^{2s_i} = (Norm(h_2)) \prod_{i=1}^{k_1} \mathfrak{m}_i^{4s_i} \prod_{i=k_1+1}^{k_2} \mathfrak{m}_i^{2s_i}.$$

Note that $Norm(h_1)$ and $Norm(h_2)$ are both sums of squares. Hence, we get the desired result. □

3 Vanishing of the top Chern class : Theorem A

In this section, we assume the structure theorem **Theorem B** and give a quick sketch of **Theorem A**.

We recall the setup once again. Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a real closed field $\mathbf{R}$. Assume further that the set $X(\mathbf{R})$ of real points is not empty, hence infinite. Let L be a projective A-module of rank 1. We denote $K_A = \wedge^n(\Omega_{A/\mathbf{R}})$ by K. Assume that $X(\mathbf{R})$ has precisely t closed and bounded components.

Let $\bar{\mathbf{R}}$ denote the algebraic closure of $\mathbf{R}$.Let $A_{\bar{\mathbf{R}}} = A \otimes_{\mathbf{R}} \bar{\mathbf{R}}$ and $\bar{X} = \text{Spec}(A_{\bar{\mathbf{R}}})$. Let $\pi : \bar{X} \to X$ be the canonical map and $\pi_* : \text{CH}_0(\bar{X}) \to \text{CH}_0(X)$. Let $G = \pi_*(\text{CH}_0(\bar{X}))$. Then, G is divisible and torsion-free (refer to [4, Section 4] and [1, Section 4] for more details). Let $\text{E}^{\bar{\mathbf{R}}}(L)$ be the kernel of the surjection $\text{E}(A, L) \twoheadrightarrow \text{E}(\mathbf{R}(A), \mathbf{R}(L))$. Then, there is a natural surjective map $\text{E}^{\bar{\mathbf{R}}}(L) \twoheadrightarrow G$

and hence we get the following commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{E}^{\bar{\mathbf{R}}}(L) & \longrightarrow & \mathrm{E}(A,L) & \xrightarrow{\Gamma_L} & \mathrm{E}(\mathbf{R}(A),\mathbf{R}(L)) & \longrightarrow & 0 \\ & & {\scriptstyle\Psi_L}\big\downarrow & & {\scriptstyle\Theta_L}\big\downarrow & & {\scriptstyle\Phi_L}\big\downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & \mathrm{CH}_0(X) & \longrightarrow & \mathrm{CH}_0(X)/G & \longrightarrow & 0 \end{array} \qquad (*)$$

We quote two theorems pertaining to (*). The first theorem is due to Colliot-Thélène and Scheiderer about $\mathrm{CH}_0(X)/G$.

Theorem 3.1. *[6, Theorem 1.3(d)] Assume that $X(\mathbf{R})$ has precisely t closed and bounded components. Then, $\mathrm{CH}_0(X)/G$ is a vector space of dimension t over the field $\mathbb{Z}/(2)$.*

Theorem 3.2. *[4, Theorem 3.1] Ψ_L is an isomorphism in diagram (*)*

As a consequence, we obtain :

Corollary 3.3.

$$\mathrm{E}_0(\mathbf{R}(A),\mathbf{R}(L)) \xrightarrow{\sim} \mathrm{CH}_0(X)/G$$

and hence

$$\mathrm{E}_0(A,L) \twoheadrightarrow \mathrm{CH}_0(X)$$

is an isomorphism.

Proof. We know that $\mathrm{E}_0(\mathbf{R}(A),\mathbf{R}(L))$ is a vector space of rank $\leq t$ from the structure theorem **Theorem B** and (2.16). But $\mathrm{E}_0(\mathbf{R}(A),\mathbf{R}(L)) \twoheadrightarrow \mathrm{CH}_0(X)/G$ and by (3.1), $\mathrm{CH}_0(X)/G$ is a vector space of rank t. Hence, so is $\mathrm{E}_0(\mathbf{R}(A),\mathbf{R}(L))$ and

$$\mathrm{E}_0(\mathbf{R}(A),\mathbf{R}(L)) \xrightarrow{\sim} \mathrm{CH}_0(X)/G.$$

So using this relation and (3.2) in the diagram (*), we get an induced diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {\mathrm{E}_0}^{\bar{\mathbf{R}}}(L) & \longrightarrow & \mathrm{E}_0(A,L) & \longrightarrow & \mathrm{E}_0(\mathbf{R}(A),\mathbf{R}(L)) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\wr} & & \big\downarrow & & \big\downarrow{\scriptstyle\wr} & & \\ 0 & \longrightarrow & G & \longrightarrow & \mathrm{CH}_0(X) & \longrightarrow & \mathrm{CH}_0(X)/G & \longrightarrow & 0 \end{array} \qquad (**)$$

Therefore, using the 5-lemma, we get $\mathrm{E}_0(A,L) \xrightarrow{\sim} \mathrm{CH}_0(X)$. $\square$

We now give a proof of **Theorem A**, which we recall below.

Theorem 3.4. *Let* $\mathbf{R}$ *be a real closed field. Let* $X = \mathrm{Spec}(A)$ *be a smooth affine variety of dimension* $n \geq 2$ *over* $\mathbf{R}$. *Let* $X(\mathbf{R})$ *denote the* $\mathbf{R}$*-rational points of the variety. Let* K *denote the module* $\wedge^n(\Omega_{A/\mathbf{R}})$. *Let* P *be a projective* A*-module of rank* n *and let* $\wedge^n(P) = L$. *Assume that* $C_n(P) = 0$ *in* $\mathrm{CH}_0(X)$. *Then* $P \simeq A \oplus Q$ *in the following cases:*

1. $X(\mathbf{R})$ *has no closed and bounded semialgebraically connected component.*
2. *For every closed and bounded semialgebraically connected component* W *of* $X(\mathbf{R})$, $L_W \not\simeq_{sa} K_W$ *where* K_W *and* L_W *denote restriction of (induced) line bundles on* $X(\mathbf{R})$ *to* W.
3. n *is odd.*

Moreover, if n *is even and* L *is a rank* 1 *projective* A*-module such that there exists a closed and bounded semialgebraically connected component* W *of* $X(\mathbf{R})$ *with the property that* $L_W \simeq K_W$, *then there exists a projective* A*-module* P *of rank* n *such that* $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ *(hence* $C_n(P) = 0$*) but* P *does not have a free summand of rank* 1.

Proof. We note that due to (3.2), the diagram (*) can be re-written as:

$$
\begin{array}{ccccccccc}
 & & & & \ker(\Theta_L) & \xrightarrow{\sim} & \ker(\Phi_L) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{E}^{\bar{\mathbf{R}}}(L) & \longrightarrow & \mathrm{E}(A, L) & \xrightarrow{\Gamma_L} & \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L)) & \longrightarrow & 0 \\
 & & {\scriptstyle \Psi_L} \downarrow \wr & & {\scriptstyle \Theta_L} \downarrow & & {\scriptstyle \Phi_L} \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & \mathrm{CH}_0(X) & \longrightarrow & \mathrm{CH}_0(X)/G & \longrightarrow & 0
\end{array}
$$

Hence, using (3.1) and the structure theorem **Theorem B**, we get that $\ker(\Phi_L)$ ($\simeq \ker(\Theta_L)$) is a free abelian group of rank r where r denotes the number of closed and bounded components C_i of $X(\mathbf{R})$ with the property that $L_{C_i} \simeq K_{C_i}$.

Let P be a projective A-module of rank n with $\wedge^n(P) \simeq L$ and let $\chi : L \to \wedge^n(P)$ be an L-orientation of P. Then, $\Theta_L(e(P, \chi)) = C_n(P)$. In view of (2.17), to prove the theorem, it is enough to prove that $C_n(P) = 0 \Rightarrow e(P, \chi) = 0$.

Proof in cases 1. and 2.
Note that in this case, by the structure theorem of $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$, Φ_L is an isomorphism and hence, Θ_L is an isomorphism.

Proof in case 3. When n is odd, there is an automorphism Δ of P with determinant -1. Let $\alpha : P \twoheadrightarrow I$ where I is a finite intersection of maximal ideals. Let ω_I be a local L-orientation of I induced by α. Using this and Δ, we get that $2e(P,\chi) = (I,\omega_I) + (I,-\omega_I)$. Since the canonical map $\mathrm{E}_0(A,L) \twoheadrightarrow \mathrm{CH}_0(X)$ is an isomorphism by (3.3) and $(I) \mapsto C_n(P)$, we have $(I) = 0$ in $\mathrm{E}_0(A,L)$. Therefore, by [1, Proposition 3.7], $2e(P,\chi) = (I,\omega_I) + (I,-\omega_I) = 0$. Since $C_n(P) = 0$, $e(P,\chi) \in \ker(\Theta_L)$, which is a free abelian group. Hence, $e(P,\chi) = 0$.

Finally, let n be even, and $L_W \simeq K_W$ for some closed and bounded semialgebraically connected component W. Then, using the structure theorem **Theorem B**, $\ker(\Theta_L) \simeq \ker(\Phi_L) \neq 0$. Since $\mathrm{E}_0(A,L) \to \mathrm{CH}_0(X)$ is an isomorphism, by abuse of notation, we denote the canonical map $\mathrm{E}(A,L) \to \mathrm{E}_0(A,L)$ by Θ_L. Then, as in [2, Lemma 3.3], there exists a reduced ideal J of height n such that J is a surjective image of $L \oplus A^{n-1}$ and a local L-orientation ω_J which is not a global orientation, i.e. $(J,\omega_J) \neq 0$ in $\mathrm{E}(A,L)$. Since n is even, as in [2, Lemma 3.6], we can get a rank n projective module P, which is stably isomorphic to $L \oplus A^{n-1}$ (i.e. $P \oplus A \simeq L \oplus A^{n-1} \oplus A$) and $\chi : L \xrightarrow{\sim} \wedge^n(P)$ such that $e(P,\chi) = (I,\omega_I) \neq 0$ in $\mathrm{E}(A,L)$. Note that since P is stably isomorphic to $L \oplus A^{n-1}$, $C_n(P) = 0$ but $e(P,\chi) \neq 0$ and hence, by (2.17), $P \not\simeq Q \oplus A$. This completes the proof. □

4 Elementary Paths in $X(\mathbf{R})$: Technical Lemma

Let $X = \mathrm{Spec}(A)$ be a smooth affine variety over $\mathbf{R}$ of dimension $n \geq 2$. Assume that the set $X(\mathbf{R})$ of real points is not empty, hence infinite. In this section, we analyse elementary paths in $X(\mathbf{R})$.

Let γ be a non-degenerate elementary path in $X(\mathbf{R})$ as defined in (2.10). Then the Zariski closure $\bar{\gamma}$ in X is an irreducible curve. Let $\mathfrak{p}$ be the prime ideal of A defining this curve and $B = A/\mathfrak{p}$. Let C be the normalisation of B. Then $Z = \mathrm{Spec}(C)$ is a smooth curve. Let $\Sigma_B = \{1 + \sum f_i^2 | f_i \in B\}$. Then $B' = \Sigma_B^{-1} C$ is the normalisation of $\mathbf{R}(B)$ and $B' \hookrightarrow \mathbf{R}(C)$. B' contains all the real maximal ideals of C and only finitely many non-real maximal ideals (which contract to the singularities of $\mathbf{R}(B)$). In particular, that means $\mathbf{R}(C) = \mathbf{R}(B')$.

Using this, we get :

Lemma 4.1. *Let $\mathfrak{m}$ be a maximal ideal of B'. If $\mathfrak{m}$ is non-real, then $\mathfrak{m}$ is principal. If $\mathfrak{m}$ is real, then $\mathfrak{m}^2$ is a principal ideal, generated by a sum of squares.*

Proof. Let $\mathfrak{c} = \mathfrak{c}_{B'/\mathbf{R}(B)}$ be the conductor of B' over $\mathbf{R}(B)$. Now consider the Mayer-Vietoris sequence corresponding to the conductor,

$$U(\frac{B'}{\mathfrak{c}}) \to \mathrm{Pic}(\mathbf{R}(B)) \to \mathrm{Pic}(B') \oplus \mathrm{Pic}(\frac{\mathbf{R}(B)}{\mathfrak{c}}) \to \mathrm{Pic}(\frac{B'}{\mathfrak{c}}).$$

Since $\mathfrak{c}$ has height 1, $\mathrm{Pic}(\mathbf{R}(B)/\mathfrak{c}) = \mathrm{Pic}(B'/\mathfrak{c}) = 0$. Hence, we get $\mathrm{Pic}(\mathbf{R}(B)) \twoheadrightarrow \mathrm{Pic}(B')$. Since $\mathbf{R}(B)$ contains only real maximal ideals, by (2.6) $\mathrm{Pic}(\mathbf{R}(B))$ is 2-torsion. Hence, $\mathrm{Pic}(B')$ is also 2-torsion. Putting this together with (2.20), we get that every non-real maximal ideal of B' is principal.

Let $\mathfrak{m}$ be a real maximal ideal of B'. Since B' is a localisation of a smooth, affine curve over $\mathbf{R}$, (2.6) gives us $\mathfrak{m}^2\mathbf{R}(B') = (\sum c_i^2)$ where $\mathfrak{m} = (c_1, c_2, \ldots, c_n)$. B' has only finitely many non-real maximal ideals, say $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_k$. Since they are all principal, let $\prod_{i=1}^k \mathfrak{d}_i = (x)$. Further, choose an element $y \in \mathfrak{m}^2 \backslash (\cup_{i=1}^k \mathfrak{d}_i)$. Then consider the element $z = x^2(\sum c_i^2) + y^2$. This element clearly does not belong to any real maximal ideal $\mathfrak{m}'$ other than $\mathfrak{m}$ since

$$z \in \mathfrak{m}' \Rightarrow xc_i \in \mathfrak{m}' \Rightarrow c_i \in \mathfrak{m}' \Rightarrow \mathfrak{m} \subseteq \mathfrak{m}'.$$

Also the choice of y means that $z \notin \mathfrak{d}_i$. Locally, z generates $\mathfrak{m}^2$ hence we have $\mathfrak{m}^2 = (x^2(\sum c_i^2) + y^2)$ which proves the lemma. □

This in turn gives us :

Lemma 4.2. *Every non-real maximal ideal $\mathfrak{d}$ of B' is generated by a function which is positive at all real points, i.e. $\mathfrak{d} = (h)$ and $h = h_2/h_1$ where $h_1, h_2 \in B'$ and both are sums of squares in B'.*

Proof. Since $\mathfrak{d}$ is principal, let $\mathfrak{d} = (x)$. Then by (2.20), we know that there exist f and g which are sums of squares such that $(\mathfrak{d} \cap C) \prod \mathfrak{m}_j^2(f) = \prod \mathfrak{m}_i^2(g)$. Hence, we get $\mathfrak{d}\prod(\mathfrak{m}_j B')^2(f) = \prod(\mathfrak{m}_i B')^2(g)$. Hence, by (4.1), we get that $(x)(h_1) = (h_2)$ where h_1 and h_2 are sums of squares in B'. This means there exists a unit $u \in B'$ such that $uxh_1 = h_2$. Putting $h = ux$, we obtain the result. $\square$

Remark 4.3. Since non-real points of B' are principal, $\mathrm{Pic}(B') \xrightarrow{\sim} \mathrm{Pic}(\mathbf{R}(B'))$.

From (2.7), there exists a generator χ of $\Omega_{\mathbf{R}(C)/\mathbf{R}}$ which we fix through the rest of the argument. We summarise the information in the commutative diagrams below :

$$\begin{array}{ccccc} B & \xhookrightarrow{\xi} & C & & \\ \downarrow & & \downarrow & & \\ \mathbf{R}(B) & \hookrightarrow & B' = \Sigma_B^{-1}C & \hookrightarrow & \mathbf{R}(C) \end{array} \qquad (**\!*1)$$

which along with the definition of elementary path gives us :

$$\begin{array}{ccccc} \gamma & \hookrightarrow & (\mathrm{Spec}(B))(\mathbf{R}) & \xrightarrow{\sim} & Max(\mathbf{R}(B)) \\ \wr\uparrow & & \uparrow & & \uparrow\xi^* \\ [P,Q] & \hookrightarrow & (\mathrm{Spec}(B'))(\mathbf{R}) & \hookrightarrow & Max(B') \\ & & \wr\downarrow & & \downarrow \\ & & (\mathrm{Spec}(C))(\mathbf{R}) & \hookrightarrow & Max(C) \end{array} \qquad (**\!*2)$$

Consider $\xi^*(P)$ and $\xi^*(Q)$ which are elements of $Max(\mathbf{R}(B)) \hookrightarrow X(\mathbf{R})$ and let $\mathcal{M}_0$ and $\mathcal{M}_1$ be the corresponding maximal ideals of $\mathbf{R}(A)$. Then $\mathcal{M}_0$ is the starting point of γ and $\mathcal{M}_1$ is the endpoint. Let $\mathfrak{m}_0, \mathfrak{m}_1$ be the maximal ideals of $\mathbf{R}(B)$ corresponding to $\mathcal{M}_0$ and $\mathcal{M}_1$. Let $\mathfrak{m}_P$ and $\mathfrak{m}_Q$ be the maximal ideals of B' corresponding to P and Q respectively. Recall that (2.8) gave us a function $f_{P,Q} \in \mathbf{R}(C)$ with special properties.

Lemma 4.4. *We can choose $g_{P,Q}$ in B' such that :*

1. $(g_{P,Q}) = \mathfrak{m}_P \cap \mathfrak{m}_Q$ *in* B'.
2. *if* $dg_{P,Q} = t\ \chi$, *then it has opposite orientations at both points, i.e.* $sign(t(P)) = -1, sign(t(Q)) = 1$.
3. $g_{P,Q}$ *is positive at all points outside the closed interval* $]P, Q[$.

Proof. Since $\mathbf{R}(C) \xrightarrow{\sim} \mathbf{R}(B')$, using (2.8) there exists a function $f_{P,Q} \in \mathbf{R}(B')$ satisfying $(f_{P,Q}) = \mathfrak{m}_P\mathbf{R}(B') \cap \mathfrak{m}_Q\mathbf{R}(B')$ and $sign(g(P)) = -1, sign(g(Q)) = 1$ where $df_{P,Q} = g\ \chi$. Further, $f_{P,Q}$ was positive outside the closed interval $[P, Q]$.

Since $\mathbf{R}(B')$ is a localisation of B', we have $f_{P,Q} = f/u$ where $u = 1+\sum a_i^2$ and $f, u, a_i \in B'$. Note that since u is a sum of squares, $sign(f_{P,Q}(R)) = sign(f(R))$ for all points $R \in (\operatorname{Spec} B')(\mathbf{R})$. Let $df = g_1\chi$. Since $f_{P,Q}(P) = f_{P,Q}(Q) = 0$, we have $g_1(P) = u(P)g(P)$ and $g_1(Q) = u(Q)g(Q)$ and hence, $sign(g_1(P)) = -1$ and $sign(g_1(Q)) = 1$ so they continue to have opposite orientations.

By (4.3), $\exists h \in B'$ such that $\mathfrak{m}_P \cap \mathfrak{m}_Q = (h)$. Denote the non-real maximal ideals of B' by $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_k$. By (4.2), $\exists f_i, 1 \leq i \leq k$ such that $\mathfrak{d}_i = (f_i)$ where f_i is positive at all real points of B'. Then, $f = u_1 \prod_{i=1}^k f_i^{r_i} h$ where u_1 is a unit in B'. Let $g_{P,Q} = u_1 h$. Since u_1 is a unit, $(g_{P,Q}) = \mathfrak{m}_P \cap \mathfrak{m}_Q$. Moreover we have $f = \prod_{i=1}^k f_i^{r_i} g_{P,Q}$. Hence, $sign(f(R)) = sign(g_{P,Q}(R)) \forall R \in (\operatorname{Spec}(B'))(\mathbf{R})$. $g_{P,Q}$ is positive at all points outside the closed interval $]P, Q[$. Also, if $dg_{P,Q} = t\chi$, then $sign(t(P)) = sign(g_1(P)) = -1$ and $sign(t(Q)) = sign(g_1(Q)) = 1$ as argued previously. This completes the proof. □

Now we prove a lemma which says that if γ contains only non-singular points of $(\operatorname{Spec}(B))(\mathbf{R})$, then $\mathfrak{m}_0 \cap \mathfrak{m}_1$ is a complete intersection in $\mathbf{R}(B)$. In what follows, for $T \in (\operatorname{Spec}(B))(\mathbf{R})$, we denote the corresponding maximal ideal of $\mathbf{R}(B)$ by $\mathfrak{m}_T$.

Lemma 4.5. *Suppose γ contains only non-singular points of* $(\operatorname{Spec}(B))(\mathbf{R})$. *Then given a finite set $\{T_1, \ldots, T_r\} \bigcup \{T'_1, \ldots, T'_s\}$ of points in* $(\operatorname{Spec}(B))(\mathbf{R})$ *not contained in γ, there exists $g \in \mathbf{R}(B)$ such that*

1. $(g) = \mathfrak{m}_0 \cap \mathfrak{m}_1$
2. $g - 1 \in J$ *where* $J = (\cap_{i=1}^r \mathfrak{m}_{T_i}) \cap (\cap_{j=1}^s \mathfrak{m}_{T'_j}) \cap \mathfrak{c}_{B'/\mathbf{R}(B)}$
3. $sign(dg/\chi) = -1$ *at* $\mathfrak{m}_0$ *and* $sign(dg/\chi) = 1$ *at* $\mathfrak{m}_1$.

Proof. Note that without loss of generality, we can assume that points T_i and T'_j correspond to smooth maximal ideals of $\mathbf{R}(B)$, and hence T_i, T'_j can be regarded as points of $(\operatorname{Spec}(B'))(\mathbf{R})$ as well (since $(\xi^*)^{-1}(T_i)$ and $(\xi^*)^{-1}(T'_j)$ are singleton). Consider the function $g_{P,Q} \in B'$ as in (4.4) and any point $T \in (\operatorname{Spec}(B'))(\mathbf{R})$ such that the corresponding maximal ideal $\mathfrak{m}_T$ contains JB'. Then either it is T_i

or T'_j or it is in the support of the conductor ideal $\mathfrak{c}_{B'/\mathbf{R}(B)}$. Since none of these points are contained in $[P, Q]$ (since none of the images under ξ^* are contained in γ), we have that $g_{P,Q}(T) > 0$. Consider $\bar{g}_{P,Q} \in (B'/J)^*$. Then, applying (2.18) (with $I = \sqrt{J}/J$), we get $g_1 \in B'$ such that $\bar{g_1}^2 = \bar{g}_{P,Q} \in (B'/J)^*$ and $(g_1) + J = B'$. Hence, there exists $a \in B', x \in J$ such that $ag_1 + x = 1 \in B'$. Note that $(g_1, x) = B'$. Consider $u = g_1^2 + x^2$. Let $\mathfrak{m}$ be a maximal ideal of B'. If $J \subseteq \mathfrak{m}$, then $x \in \mathfrak{m}$. Therefore, $g_1 \notin \mathfrak{m}$ and hence $u \notin \mathfrak{m}$. If $J \not\subseteq \mathfrak{m}$, then $\mathfrak{c}_{B'/\mathbf{R}(B)} \not\subseteq \mathfrak{m}$ and hence if $\mathfrak{n} = \mathfrak{m} \cap \mathbf{R}(B)$, then $\mathbf{R}(B)_{\mathfrak{n}} = B'_{\mathfrak{m}}$. As $\mathbf{R}(B)$ has only real maximal ideals, we get that $\mathfrak{m}$ is a real maximal ideal of B'. Hence, $u \in \mathfrak{m} \Rightarrow g_1, x \in \mathfrak{m} \Rightarrow 1 \in \mathfrak{m}$ which is a contradiction. Therefore, u is a unit in B' and since it is a sum of squares, it is positive at all real points. Consider $g = u^{-1}g_{P,Q}$. Therefore, with bar denoting elements in B'/J,

$$\bar{g} = \overline{u^{-1}}\bar{g}_{P,Q} = \bar{u}^{-1}\bar{g}_{P,Q} = \bar{g_1^2}^{-1}\bar{g}_{P,Q} = \bar{g_1}^{-2}\bar{g_1}^2 = 1 \in B'/J.$$

Hence, $g = 1 + y$ for some element $y \in J \subseteq \mathfrak{c}_{B'/\mathbf{R}(B)}$. Hence, using (2.19), we get $\mathfrak{m}_0 \cap \mathfrak{m}_1 = (g)$. Further, we see that at the points $T_i, T'_j, g(T_i) = g(T'_j) = 1$ and $g(T) = 1$ for each point $T \in (\mathrm{Spec}(B))(\mathbf{R})$ which is a singular point.

Note that

$$\frac{\Omega_{\mathbf{R}(B)/\mathbf{R}}}{\mathfrak{m}_0\Omega_{\mathbf{R}(B)/\mathbf{R}}} \xrightarrow{\sim} \frac{\Omega_{B'/\mathbf{R}}}{\mathfrak{m}_P\Omega_{B'/\mathbf{R}}}$$

and

$$\frac{\Omega_{\mathbf{R}(B)/\mathbf{R}}}{\mathfrak{m}_1\Omega_{\mathbf{R}(B)/\mathbf{R}}} \xrightarrow{\sim} \frac{\Omega_{B'/\mathbf{R}}}{\mathfrak{m}_Q\Omega_{B'/\mathbf{R}}}$$

Since $g = u^{-1}g_{P,Q}$ in B', we have $dg = u^{-1}dg_{P,Q} + g_{P,Q}d(u^{-1})$ in $\Omega_{B'/\mathbf{R}}$. Hence,

$$(dg/\chi)(P) = u^{-1}(P)(dg_{P,Q}/\chi)(P), \qquad (dg/\chi)(Q) = u^{-1}(Q)(dg_{P,Q}/\chi)(Q).$$

Therefore, $sign((dg/\chi)(P)) = -1$ and $sign((dg/\chi)(Q)) = 1$. □

Let L be a projective A-module of rank 1. We prove the final lemma of this section which shows that if γ contains only non-singular points of $\mathbf{R}(B)$, then there exists $\beta : \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-1} \twoheadrightarrow \mathcal{M}_0 \cap \mathcal{M}_1$.

We recall the set-up once again. Recall that $B = A/\mathfrak{p} = \bar{\gamma}^{Zar}$ where γ is an elementary path in $X(\mathbf{R})$ with starting point $\mathcal{M}_0$ and endpoint $\mathcal{M}_1$. Let C be the normalisation of B, B' be the normalisation of $\mathbf{R}(B)$ and hence $\mathbf{R}(B') = \mathbf{R}(C)$. χ is a generator of $\Omega_{\mathbf{R}(C)/\mathbf{R}} = \Omega_{\mathbf{R}(B')/\mathbf{R}}$. Let $\mathfrak{c}_{B'/\mathbf{R}(B)}$ be the conductor ideal of B' over $\mathbf{R}(B)$. Let $\mathfrak{C} \supseteq \mathfrak{p}\mathbf{R}(A)$ be the ideal of $\mathbf{R}(A)$ such that $\mathfrak{C}/\mathfrak{p}\mathbf{R}(A) = \mathfrak{c}_{B'/\mathbf{R}(B)}$.

Since A is smooth, $\mathfrak{p}\mathbf{R}(A)_{\mathfrak{p}} = (a_1, a_2, \ldots, a_{n-1})$ with $a_i \in \mathbf{R}(A)$. Let $B_1 = \mathbf{R}(A)/(a_1, a_2, \ldots, a_{n-1})$. Note that $(a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p}\mathbf{R}(A) \cap I$ for some ideal I of $\mathbf{R}(A)$ not contained in $\mathfrak{p}\mathbf{R}(A)$.

Hence onward we use the following convention. For a maximal ideal $\mathcal{M}$ of $\mathbf{R}(A)$ containing $\mathfrak{p}\mathbf{R}(A)$, $\overline{\mathcal{M}}$ denotes the corresponding maximal ideal of B_1 and $\mathfrak{m}$ will denote the corresponding maximal ideal of $\mathbf{R}(B)$, i.e. $\overline{\mathcal{M}} = \mathcal{M}B_1$ and $\mathfrak{m} = \mathcal{M}\mathbf{R}(B)$. Since $\mathfrak{p}\mathbf{R}(A) + I$ is of height n in $\mathbf{R}(A)$, there are finitely many maximal ideals containing $\mathfrak{p}\mathbf{R}(A) + I$. Let those be denoted by $T_1, \ldots, T_r$ in $X(\mathbf{R})$.

Lemma 4.6. *Suppose, with the assumptions and notations as above, there exists $f \in \mathbf{R}(A)$ such that :*

1. $f \in \mathfrak{C} \cap (\mathfrak{p}\mathbf{R}(A) + I)$.
2. $\operatorname{Spec}(\mathbf{R}(A)/f) \cap \gamma = \emptyset$.

Then, there exists $a_n \in \mathbf{R}(A)$ such that $(a_1, a_2, \ldots, a_n) = \mathcal{M}_0 \cap \mathcal{M}_1$, $a_n - 1 \in (a_1, a_2, \ldots, a_{n-1}, f)$ and sign$((d(\bar{a_n})/\chi)(P)) = -1$ and sign$((d(\bar{a_n})/\chi)(Q)) = 1$ where $\bar{a_n}$ denotes the image of a_n under the map $\mathbf{R}(A) \to \mathbf{R}(C)$. Moreover, if there exists $\tau \in L$ such that $\mathbf{R}(L)_f = \tau\mathbf{R}(L)_f$, then there exists a surjection $\beta : \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-1} \twoheadrightarrow \mathcal{M}_0 \cap \mathcal{M}_1$ such that $\beta(\tau) = a_1 + ha_n$ and $\beta(e_i) = a_i$; $2 \leq i \leq n$ where $(e_2, \cdots, e_n)$ denotes a basis of $\mathbf{R}(A)^{n-1}$.

Proof. Since $\operatorname{Spec}(\mathbf{R}(A)/f) \cap \gamma = \emptyset$, we have $f \notin \mathfrak{p}$.

$$\text{Let} \quad \Upsilon = \{T \in X(\mathbf{R}) | \mathfrak{p} + f \in \mathcal{M}_T\} = V((\mathfrak{p} + f)\mathbf{R}(A)),$$

$$\{T'_1, \ldots, T'_s\} = \Upsilon \setminus V(\mathfrak{C} \cap (\mathfrak{p}\mathbf{R}(A) + I)),$$

$$\{T_1, \ldots, T_r\} = V(\mathfrak{p}\mathbf{R}(A) + I) \setminus V(\mathfrak{C}) \quad \text{and}$$

$$J = (\mathfrak{p}\mathbf{R}(A) + I) \cap \mathfrak{C} \cap (\cap_{j=1}^{s} \mathcal{M}_{T'_j}) \subset \mathbf{R}(A).$$

$$\text{Then} \quad \sqrt{J} = (\cap_{i=1}^{r} \mathcal{M}_{T_i}) \cap \sqrt{\mathfrak{C}} \cap (\cap_{j=1}^{s} \mathcal{M}_{T'_j}) = \bigcap_{T \in \Upsilon} \mathcal{M}_T = \sqrt{\mathfrak{p}\mathbf{R}(A) + (f)}.$$

Note that since $\mathfrak{p}B_1 \cap IB_1 = 0$, the above equality implies

$$\sqrt{IB_1} \cap \sqrt{\mathfrak{C}B_1} \cap (\cap_{j=1}^{s} \overline{\mathcal{M}_{T_j'}}) \subseteq \sqrt{(f)B_1}.$$

Moreover, as $\gamma \subset V(\mathfrak{p}\mathbf{R}(A))$ and $\mathrm{Spec}(\mathbf{R}(A)/(f)) \cap \gamma = \emptyset$, $\gamma \cap \Upsilon = \emptyset$. By (4.5), there exists $g \in \mathbf{R}(B)$ such that

- $(g) = \mathfrak{m}_0 \cap \mathfrak{m}_1$
- $g - 1 \in \sqrt{J\mathbf{R}(B)} = \sqrt{J}/\mathfrak{p}\mathbf{R}(\mathbf{A})$
- as an element of $\mathbf{R}(C)(\supseteq \mathbf{R}(B))$, $sign(dg/\chi) = -1$ at $\mathfrak{m}_P(= \mathfrak{m}_0\mathbf{R}(C))$ and $sign(dg/\chi) = 1$ at $\mathfrak{m}_Q(= \mathfrak{m}_1\mathbf{R}(C))$.

Consider the ring $B_1' = \mathbf{R}(B) \oplus \mathbf{R}(A)/I$. There is a natural surjection $B_1' \twoheadrightarrow \mathbf{R}(B)$ and through this, we have $(\mathfrak{m}_0 \cap \mathfrak{m}_1) \oplus \mathbf{R}(A)/I = ((g, 1))$ in B_1'. The natural map $B_1 \twoheadrightarrow \mathbf{R}(B)$ factors through as $B_1 \hookrightarrow B_1' \twoheadrightarrow \mathbf{R}(B)$. Note that the conductor $\mathfrak{c}_{B_1'/B_1} = \overline{\mathfrak{p}\mathbf{R}(A) + I}$ in B_1 maps bijectively to $(\mathfrak{p}\mathbf{R}(A)+I)/\mathfrak{p}\mathbf{R}(A) \oplus (\mathfrak{p}\mathbf{R}(A)+I)/I$ in B_1'.

Since $g - 1 \in \sqrt{J\mathbf{R}(B)} = \sqrt{J}/\mathfrak{p}\mathbf{R}(\mathbf{A})$, the equation $Y^2 - g$ has a solution in $\mathbf{R}(B)/\sqrt{J}\mathbf{R}(B)$. Hence, by (2.18), there exists $g_1 \in \mathbf{R}(B)$ such that $g_1^2 - g \in J\mathbf{R}(B)$. As $(g) + J\mathbf{R}(B) = \mathbf{R}(B)$, $(g_1) + J\mathbf{R}(B) = \mathbf{R}(B)$. Let $y \in J$ such that $(g_1) + (y) = \mathbf{R}(B)$. Then, $v = g_1^2 + y^2$ is a unit in $\mathbf{R}(B)$. Consider the element $g_2 = v^{-1}g$. Then, $(g_2) = \mathfrak{m}_0 \cap \mathfrak{m}_1$ and $g_2 - 1 \in J\mathbf{R}(B)$. Further, since v is a sum of squares and a unit, we get that

$$sign(\frac{dg_2}{\chi}(P)) = sign(\frac{dg}{\chi}(P)) = -1 \quad \text{and}$$

$$sign(\frac{dg_2}{\chi}(Q)) = sign(\frac{dg}{\chi}(Q)) = 1.$$

Note that since $\mathfrak{p}\mathbf{R}(A) \subsetneqq J \subseteq \mathfrak{p}\mathbf{R}(A) + I$ and $g_2 - 1 \in J\mathbf{R}(B) = J/\mathfrak{p}\mathbf{R}(A)$, the element $(g_2, 1) - (1, 1)$ of $\mathbf{R}(B) \oplus \mathbf{R}(A)/I(= B_1')$ belongs to the conductor ideal $\mathfrak{c}_{B_1'/B_1} = (\mathfrak{p}\mathbf{R}(A) + I)/\mathfrak{p}\mathbf{R}(A) \oplus (\mathfrak{p}\mathbf{R}(A) + I)/I \subseteq B_1'$. Hence, there exists $b \in B_1$ such that $b \mapsto (g_2, 1)$. We note that the hypothesis implies, $\overline{\mathcal{M}_0} \cap \overline{\mathcal{M}_1} \not\supseteq \bar{I}$. Hence, $(\overline{\mathcal{M}_0} \cap \overline{\mathcal{M}_1})B_1' = (\mathfrak{m}_0 \cap \mathfrak{m}_1) \oplus \mathbf{R}(A)/I = (g_2, 1)B_1'$ and therefore, by (2.19), $(b) = \overline{\mathcal{M}_0} \cap \overline{\mathcal{M}_1}$. As above, we have

$$sign(\frac{d(\tilde{b})}{\chi}(P)) = sign(\frac{dg_2}{\chi}(P)) = -1 \quad \text{and}$$

$$sign(\frac{d(\tilde{b})}{\chi}(Q)) = sign(\frac{dg_2}{\chi}(Q)) = 1$$

where $\tilde{b}$ is the image of b in $\mathbf{R}(C)$.

Since $\bar{f} \notin \overline{\mathcal{M}_0} \cap \overline{\mathcal{M}_1}$, $(\bar{f})B_1 + (b)B_1 = B_1$. Further,

$$b - 1 \mapsto (g_2 - 1, 0) \in \sqrt{J\mathbf{R}(B)} \oplus \mathbf{R}(A)/I$$

$$\Rightarrow b - 1 \in \sqrt{IB_1} \cap \mathfrak{C}B_1 \cap (\cap_{j=1}^{s} \overline{\mathcal{M}_{T'_j}}) \subseteq \sqrt{fB_1}.$$

Hence, the equation $Y^2 - b$ has a solution in $B_1/\sqrt{fB_1}$ and therefore, by (2.18), we have $a \in B_1$ such that $a^2 - b \in fB_1$. As $b \equiv 1 \; mod \sqrt{fB_1}$, $(a, \bar{f}) = B_1$ where $\bar{f}$ is the image of f in B_1. Therefore, $u = a^2 + \bar{f}^2$ is a unit in B_1. Consider $c \in \mathbf{R}(A)$ such that $\bar{c} = u^{-1}b$. Then, $\bar{c} - 1 \in (\bar{f})$. Hence, $c - 1 = \alpha f + \sum_{j=1}^{n-1} \alpha_j a_j$. Let $a_n = c - \sum_{j=1}^{n-1} \alpha_j a_j$. Therefore,

$$a_n - 1 = \alpha f \Rightarrow (a_n, f) = \mathbf{R}(A).$$

Further, $(\bar{a_n}) = (\bar{c}) = \overline{\mathcal{M}_0} \cap \overline{\mathcal{M}_1}$ where "bar" denotes the image in B_1 and hence $(a_1, a_2, \ldots, a_{n-1}, a_n) = \mathcal{M}_0 \cap \mathcal{M}_1$ in $\mathbf{R}(A)$. Also,

$$sign(\frac{d(\tilde{a_n})}{\chi})(P) = sign(\frac{d(\tilde{c})}{\chi})(P) = sign(\frac{d(\tilde{b})}{\chi}(P)) = -1 \qquad and$$

$$sign(\frac{d(\tilde{a_n})}{\chi})(Q) = sign(\frac{d(\tilde{c})}{\chi})(Q) = sign(\frac{d(\tilde{b})}{\chi}(Q)) = 1$$

where "tilde" are the images in $\mathbf{R}(C)$.

Now assume that there exists $\tau \in L$ such that $\mathbf{R}(L)_f = \tau\mathbf{R}(A)_f$. Since $\mathbf{R}(A)f + \mathbf{R}(A)a_n = \mathbf{R}(A)$, we have $\mathbf{R}(L)/a_n\mathbf{R}(L) = \mathbf{R}(L)_f/a_n\mathbf{R}(L)_f$. Therefore as $\mathbf{R}(L)_f = \tau\mathbf{R}(A)_f$, $\mathbf{R}(L)/a_n\mathbf{R}(L)$ is a free $\mathbf{R}(A)/(a_n)$-module of rank 1 with $\bar{\tau}$ as a generator. Therefore, it is easy to see that there exists an $\mathbf{R}(A)$-linear map $\alpha : \mathbf{R}(L) \rightarrow \mathcal{M}_0 \cap \mathcal{M}_1$ such that $\alpha(\tau) = a_1 + a_n h$ with $h \in \mathbf{R}(A)$.

Let

$$\beta : \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-1} \rightarrow \mathcal{M}_0 \cap \mathcal{M}_1$$

be an $\mathbf{R}(A)$-linear map map defined as $\beta(l) = \alpha(l)$ for $l \in \mathbf{R}(L)$ and $\beta(e_i) = a_i$; $2 \leq i \leq n$ where $(e_2, \cdots, e_n)$ is a basis of $\mathbf{R}(A)^{n-1}$. Then, $\beta(\tau - he_n) = a_1 + ha_n - ha_n = a_1$. Hence, as $\mathcal{M}_0 \cap \mathcal{M}_1 = (a_1, a_2, \ldots, a_n)$, β is a surjection. Thus, we obtain the result. $\square$

5 Elementary Paths in $Z(\mathbf{R})$

Let $X = \mathrm{Spec}(A)$ be a smooth affine variety over $\mathbf{R}$ of dimension $n \geq 2$. Assume further that the set $X(\mathbf{R})$ of real points is not empty, hence infinite. Let L be a projective A-module of rank 1. We denote $K_A = \wedge^n(\Omega_{A/\mathbf{R}})$ by K.

Let $\mathcal{E} = L \otimes_A K$. Let $D = \oplus_{-\infty<i<\infty}\mathcal{E}^i$. Let $Z = \mathrm{Spec}(D)$. Then there is a natural map $A \hookrightarrow D$ which gives rise to a natural surjection $Z \twoheadrightarrow X$ which induces a natural map $Z(\mathbf{R}) \twoheadrightarrow X(\mathbf{R})$ which we denote by Π. Looked at in the Euclidean topology, this gives an $\mathbf{R}^*$-bundle over $X(\mathbf{R})$.

In what follows, we identify points of $Z(\mathbf{R})$ (respectively $X(\mathbf{R})$) with the corresponding real maximal ideals of D (respectively A). Recall that $\mathbf{R}(A)$ denotes the ring obtained from A by inverting all elements of the type $1 + \sum_{i=1}^{l} f_i^2; f_i \in A$ and $\mathbf{R}(L) = L \otimes_A \mathbf{R}(A)$. Let

$$Y = \{(\mathcal{M}, \omega_{\mathcal{M}}) | \mathcal{M} \in X(\mathbf{R}), \ \ \omega_{\mathcal{M}} : \frac{L}{\mathcal{M}L} \xrightarrow{\sim} \wedge^n(\frac{\mathcal{M}}{\mathcal{M}^2})\}.$$

Recall that the Euler class group $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ is a quotient of the free abelian group with generating set Y. We associate with Y the topological space $Z(\mathbf{R})$ as follows :

Let $\mathcal{M}$ be a real maximal ideal. Let

$$Y_{\mathcal{M}} = \{(\mathcal{M}, \omega_{\mathcal{M}}) | \omega_{\mathcal{M}} : \frac{L}{\mathcal{M}L} \xrightarrow{\sim} \wedge^n(\frac{\mathcal{M}}{\mathcal{M}^2})\}.$$

The differential map $d : A \to \Omega_{A/\mathbf{R}}$ induces $\wedge^n(d_{\mathcal{M}}) : \wedge^n(\mathcal{M}/\mathcal{M}^2) \xrightarrow{\sim} K/\mathcal{M}K$. Composing $\omega_{\mathcal{M}}$ with $\wedge^n(d_{\mathcal{M}})$, we get an isomorphism $\phi_{\omega_{\mathcal{M}}} : L/\mathcal{M}L \xrightarrow{\sim} K/\mathcal{M}K$. Note that

$$\frac{\mathbf{R}(\mathcal{E})}{\mathcal{M}\mathbf{R}(\mathcal{E})} = \frac{\mathbf{R}(L)}{\mathcal{M}\mathbf{R}(L)} \otimes \frac{\mathbf{R}(K)}{\mathcal{M}\mathbf{R}(K)}$$

and hence there is a natural map $\mathbf{R}(L)^2/\mathcal{M}\mathbf{R}(L)^2 \to \mathbf{R}(\mathcal{E})/\mathcal{M}\mathbf{R}(\mathcal{E})$ given by

$$\Gamma_{\omega_{\mathcal{M}}}(l \otimes l') = l \otimes \phi_{\omega_{\mathcal{M}}}(l'); \quad l, l' \in \frac{L}{\mathcal{M}L}.$$

Thus, we get :

$$\mathrm{Hom}(\frac{\mathbf{R}(L)}{\mathcal{M}\mathbf{R}(L)}, \wedge^n(\frac{\mathcal{M}}{\mathcal{M}^2})) \overset{\wedge^n(d_{\mathcal{M}})\circ}{\xrightarrow{\sim}} \mathrm{Hom}(\frac{\mathbf{R}(L)}{\mathcal{M}\mathbf{R}(L)}, \frac{\mathbf{R}(K)}{\mathcal{M}\mathbf{R}(K)}) \overset{\Gamma}{\xrightarrow{\sim}} \mathrm{Hom}(\frac{\mathbf{R}(L)^2}{\mathcal{M}\mathbf{R}(L)^2}, \frac{\mathbf{R}(\mathcal{E})}{\mathcal{M}\mathbf{R}(\mathcal{E})})$$

$$\omega_{\mathcal{M}} \mapsto \wedge^n(d_{\mathcal{M}}) \circ \omega_{\mathcal{M}} \mapsto id_{\mathbf{R}(L)} \otimes \wedge^n(d_{\mathcal{M}}) \circ \omega_{\mathcal{M}}.$$

By (2.6), $\mathbf{R}(L)^2$ is free. Let $\kappa \in L^2$ be a generator of $\mathbf{R}(L)^2$. Let $\overline{\kappa}$ denote the image of κ in $\mathbf{R}(L)^2/\mathcal{M}\mathbf{R}(L)^2$. Then $\Gamma_{\omega_{\mathcal{M}}}(\overline{\kappa})$ is a non-zero element of $\mathbf{R}(\mathcal{E})/\mathcal{M}\mathbf{R}(\mathcal{E})(= \mathcal{E}/\mathcal{M}\mathcal{E})$ and hence $\Gamma_{\omega_{\mathcal{M}}}(\overline{\kappa}) = \bar{e}$, where $e \in \mathcal{E} \setminus \mathcal{M}\mathcal{E}$. Then, this gives a map $\Theta_{\mathcal{M}} : Y_{\mathcal{M}} \to \Pi^{-1}(\mathcal{M})$, sending $(\mathcal{M}, \omega_{\mathcal{M}}) \mapsto (\mathcal{M}, e-1)$ where e is defined as above. Now, every element of $\Pi^{-1}(\mathcal{M})$ is of the form $(\mathcal{M}, e-1)$ where $e \in \mathcal{E} \setminus \mathcal{M}\mathcal{E}$. Hence, given e, we get an isomorphism sending $\bar{\kappa}$ to $\bar{e}$ and working backwards in the above diagram, we get a local orientation of $\mathcal{M}$. Hence, $\Theta_{\mathcal{M}}$ is a bijection.

Note that there is a natural action of $\mathbf{R}^*$ on $Y_{\mathcal{M}}$ and $\Pi^{-1}(\mathcal{M})$ and the above diagram shows that the map $\Theta_{\mathcal{M}}$ is compatible with the action. Putting together $\Theta_{\mathcal{M}}$ for all $\mathcal{M} \in X(\mathbf{R})$, we have $\Theta : Y \xrightarrow{\sim} Z(\mathbf{R})$. Therefore, we get a set-theoretic map $Z(\mathbf{R}) \to \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

In this section, using elementary paths in $Z(\mathbf{R})$ we show that : *image of a component of* $Z(\mathbf{R})$ *under the map* $Z(\mathbf{R}) \to \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ *is singleton.*

To show this we need to prove some auxiliary results. We first set up notations required for these results.

Suppose $f \in A$ is such that $L_f \simeq A_f \simeq K_f$. Let us fix generators τ and ρ of L_f and K_f respectively. Let $Z_f = \mathrm{Spec}(D_f)$ and $X_f = \mathrm{Spec}(A_f)$. Then $\mathcal{E}_f$ is generated by $\tau \otimes \rho$ and therefore

$$D_f = D \otimes_A A_f = A_f[T, T^{-1}];\ T = (\tau \otimes \rho).$$

With respect to the pair (τ, ρ), we assign to every $\mathcal{P} \in Z_f(\mathbf{R})$, an element of the group $\{1, -1\}$ as follows :

Definition 5.1. Let $\Theta(\mathcal{M}, \omega_{\mathcal{M}}) = \mathcal{P}$ correspond to $(\mathcal{M}, e-1)$ where $e \in \mathcal{E}, \bar{e} \neq 0 \in \mathcal{E}/\mathcal{M}\mathcal{E}$. Then if $f \notin M$, we have

$$\bar{e} = \lambda \overline{\tau} \otimes \overline{\rho} \quad : \lambda \in \mathbf{R}^*.$$

Define

$$sgn_{(\tau,\rho)}(\mathcal{P}) = sgn_{(\tau,\rho)}(\mathcal{M}, \omega_{\mathcal{M}}) = sign(\lambda).$$

Remark 5.2. Since $D_f = A_f[T, T^{-1}]$, $T = (\tau \otimes \rho)$, we can consider T as a function on $Z_f(\mathbf{R})$. Let $(\mathcal{M}, \omega_{\mathcal{M}}) \in Y$ such that $\mathcal{P} = \Theta((\mathcal{M}, \omega_{\mathcal{M}})) \in Z_f(\mathbf{R})$. Then the value of T at $\mathcal{P}$ is given by $\lambda^{-1} \in \mathbf{R}^*$ where $\bar{T} = \lambda^{-1}\bar{e} = \lambda^{-1}\Gamma_{\omega_{\mathcal{M}}}(\overline{\kappa}) \in \mathcal{E}/(\mathcal{M}\mathcal{E})$. This implies that

$$sgn_{(\tau,\rho)}(M, \omega_M) = sign(T(\mathcal{P})).$$

This further implies that if $\omega_{\mathcal{M}}, \widetilde{\omega}_{\mathcal{M}}$ are two local L-orientations of $\mathcal{M}$, then

$$sgn_{(\tau,\rho)}(M, \widetilde{\omega}_M) = sign(\alpha) sgn_{(\tau,\rho)}(M, \omega_M) \quad \text{where} \quad \widetilde{\omega}_{\mathcal{M}} = \alpha\omega_{\mathcal{M}}.$$

Since $\mathcal{E}_f$ is free, $Z_f(\mathbf{R}) \simeq X_f(\mathbf{R}) \times \mathbf{R}^+ \bigsqcup X_f(\mathbf{R}) \times \mathbf{R}^-$ where the maximal ideal $(\mathcal{M}, T-1)$ corresponds to $(\mathcal{M}, 1) \in X_f(\mathbf{R}) \times \mathbf{R}^+$. Note that $\mathcal{P} \in X_f(\mathbf{R}) \times \mathbf{R}^+$ if and only if $sgn_{(\tau,\rho)}(\mathcal{P}) = 1$.

Recall that we have a (set-theoretic) map $Z(\mathbf{R}) \to \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. For $\mathcal{P} \in Z(\mathbf{R})$, we denote its image in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ by $(\mathcal{P})$. Let Ψ be an elementary path in $Z(\mathbf{R})$ and let $\mathcal{P}$ be the starting point and $\mathcal{Q}$ be the endpoint of Ψ. Our first step is to show that under the map $Z(\mathbf{R}) \to \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$, $(\mathcal{P}) = (\mathcal{Q})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

Lemma 5.3. *Let Ψ be an elementary path in $Z(\mathbf{R})$ such that under the canonical map $\Pi : Z(\mathbf{R}) \twoheadrightarrow X(\mathbf{R})$, $\Pi(\Psi)$ is a singleton. Let $\mathcal{P} = \Theta((\mathcal{M}_0, \omega_{\mathcal{M}_0}))$ be the starting point and $\mathcal{Q} = \Theta((\mathcal{M}_1, \omega_{\mathcal{M}_1}))$ be the end point of Ψ. Then, $(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = (\mathcal{M}_1, \omega_{\mathcal{M}_1})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.*

Proof. Since $\Pi(\Psi)$ is singleton, $\mathcal{M}_0 = \mathcal{M}_1$ and $\Psi \subset \Pi^{-1}(\mathcal{M}_0)$. Choose $f \notin \mathcal{M}_0$ such that $L_f \simeq A_f \simeq K_f$. Then, $\mathcal{M}_0 = \mathcal{M}_1$ and hence, $\omega_{\mathcal{M}_0} = \lambda\omega_{\mathcal{M}_1}$. Then, choosing generators τ and ρ for L_f and K_f respectively, we can express $\Pi^{-1}(\mathcal{M}_0)$ as a union of its components $\Pi^{-1}(\mathcal{M}_0) = \mathcal{M}_0 \times \mathbf{R}^+ \bigsqcup \mathcal{M}_0 \times \mathbf{R}^-$. Since Ψ is semialgebraically connected, Ψ lies in one of them. Hence, $sgn_{(\tau,\rho)}(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = sgn_{(\tau,\rho)}(\mathcal{M}_1, \omega_{\mathcal{M}_1})$ and hence, $sign(\lambda) > 0$. But then by (2.15), this implies $(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = (\mathcal{M}_1, \omega_{\mathcal{M}_1})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$, i.e. $(\mathcal{P}) = (\mathcal{Q})$. $\square$

Now we assume that $\gamma = \Pi(\Psi)$ is not singleton and moreover that $\Pi|_\Psi : \Psi \to \gamma$ is bijective. Therefore, by [8, Theorem 3.1], γ is a nondegenerate elementary path in $X(\mathbf{R})$. We set up some notations and prove some results to deal with this case.

Let $\overline{\Psi}^{Zar} = \mathrm{Spec}(D/\mathfrak{q})$ and $\bar{\gamma}^{Zar} = \mathrm{Spec}(A/\mathfrak{p})$. Then, as γ is non-degenerate, $\mathfrak{p} = \mathfrak{q} \cap A$.

In this context, we recall the notations used in the previous section : $B = A/\mathfrak{p}A$. Let C be the normalisation of B and $\xi : B \hookrightarrow C$. Let B' be the normalisation of $\mathbf{R}(B)$ and hence $\mathbf{R}(B') = \mathbf{R}(C)$. χ is a generator of $\Omega_{\mathbf{R}(C)/\mathbf{R}} = \Omega_{\mathbf{R}(B')/\mathbf{R}}$. Since γ is an elementary path, we have an order preserving bijection $\xi^* : [P, Q] \xrightarrow{\sim} \gamma$ where $[P, Q] \subset (\mathrm{Spec}(C))(\mathbf{R})$. Then, $\xi^*(P)$ and $\xi^*(Q)$ are the start and end points of γ respectively. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be the maximal ideals of $\mathbf{R}(A)$ corresponding to $\xi^*(P)$ and $\xi^*(Q)$ respectively.

Let $\mathfrak{c}_{B'/\mathbf{R}(B)}$ be the conductor ideal of B' over $\mathbf{R}(B)$. Let $\mathfrak{C}$ be the ideal of $\mathbf{R}(A)$ containing $\mathfrak{p}\mathbf{R}(A)$ such that $\mathfrak{C}/\mathfrak{p}\mathbf{R}(A) = \mathfrak{c}_{B'/\mathbf{R}(B)}$. Since $\mathbf{R}(A)$ is regular, $\mathfrak{p}\mathbf{R}(A)_{\mathfrak{p}} = (a_1, a_2, \ldots, a_{n-1}), a_i \in \mathfrak{p}\mathbf{R}(A)$. Then, $(a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p}\mathbf{R}(A) \cap I$ for some ideal $I \subset \mathbf{R}(A)$ not contained in $\mathfrak{p}\mathbf{R}(A)$. Let $B_1 = \mathbf{R}(A)/(a_1, a_2, \ldots, a_{n-1})$.

In what follows, we shall use the following convention. For a maximal ideal $\mathcal{M}$ of $\mathbf{R}(A)$ containing $\mathfrak{p}\mathbf{R}(A)$, $\overline{\mathcal{M}}$ will denote the corresponding maximal ideal of B_1 and $\mathfrak{m}$ will denote the corresponding maximal ideal of $\mathbf{R}(B)$, i.e. $\overline{\mathcal{M}} = \mathcal{M}B_1$ and $\mathfrak{m} = \mathcal{M}\mathbf{R}(B)$.

Since $I \not\subset \mathfrak{p}\mathbf{R}(A)$, $\mathfrak{p}\mathbf{R}(A) + I$ is of height n in $\mathbf{R}(A)$, and hence $(\mathfrak{p}\mathbf{R}(A) + I) \cap \mathfrak{C}$ is an ideal of height n.

We now prove the following lemma :

Lemma 5.4. *Let $\Psi \subseteq Z(\mathbf{R})$ be a nondegenerate elementary path and $\gamma = \Pi(\Psi)$. Suppose $\Pi : \Psi \to \gamma$ is a bijection. Let $(\mathcal{M}_0, \omega_{\mathcal{M}_0})$ and $(\mathcal{M}_1, \omega_{\mathcal{M}_1})$ be such that $\Theta((\mathcal{M}_0, \omega_{\mathcal{M}_0})) = \mathcal{P}$ and $\Theta((\mathcal{M}_1, \omega_{\mathcal{M}_1})) = \mathcal{Q}$ are initial and end points of Ψ respectively. Further, assume that, with notation as above, there exists $f \in \mathbf{R}(A)$ such that :*

1. *$f \in \mathfrak{C} \cap (\mathfrak{p}\mathbf{R}(A) + I)$.*
2. *$L_f \simeq A_f \simeq K_f$.*
3. *$\mathrm{Spec}(\mathbf{R}(A)/f) \cap \gamma = \emptyset$.*

Then, $(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = (\mathcal{M}_1, \omega_{\mathcal{M}_1})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$, i.e. $(\mathcal{P}) = (\mathcal{Q})$.

Proof. Let

$$\Upsilon = \{T \in X(\mathbf{R}) | \mathfrak{p} + f \in \mathcal{M}_T\} = V((\mathfrak{p} + f)\mathbf{R}(A)),$$

$$\{T'_1, \ldots, T'_s\} = \Upsilon \setminus V(\mathfrak{C} \cap (\mathfrak{p}\mathbf{R}(A) + I)).$$

Note that

$$\sqrt{IB_1} \bigcap \overline{\mathfrak{C}} \bigcap (\cap_{j=1}^{s} \overline{\mathcal{M}_{T'_j}}) \subseteq \sqrt{(f)B_1}.$$

Also, $\gamma \subseteq X_f(\mathbf{R})$. Further, $\mathbf{R}(B)_f \xrightarrow{\sim} B'_f$ since the conductor $(\mathfrak{c}_{B'/\mathbf{R}(B)})_f$ equals the full ring B'_f. So, $\mathbf{R}(B)_f$ is regular and hence, we have the short exact sequence,

$$0 \to \frac{(a_1, a_2, \ldots, a_{n-1})\mathbf{R}(A)_f}{\mathfrak{p}(a_1, a_2, \ldots, a_{n-1})\mathbf{R}(A)_f} \to \frac{\Omega_{\mathbf{R}(A)_f/\mathbf{R}}}{\mathfrak{p}\Omega_{\mathbf{R}(A)_f/\mathbf{R}}} \to \Omega_{\mathbf{R}(B)_f/\mathbf{R}} \to 0$$

Now, since $\mathbf{R}(B)_f \xrightarrow{\sim} B'_f$, B'_f has only real maximal ideals. Hence, $B'_f \xrightarrow{\sim} \mathbf{R}(C)_f$. Hence,

$$\Omega_{\mathbf{R}(B)_f/\mathbf{R}} \xrightarrow{\sim} \Omega_{B'_f/\mathbf{R}} \xrightarrow{\sim} \Omega_{\mathbf{R}(C)_f/\mathbf{R}}.$$

Hence $\Omega_{\mathbf{R}(B)_f/\mathbf{R}}$ is generated by χ, and so the sequence is split exact. Let s be a splitting. Then $(\Omega_{\mathbf{R}(A)/\mathbf{R}}/\mathfrak{p}\Omega_{\mathbf{R}(A)/\mathbf{R}})_f$ is a free $\mathbf{R}(B)_f$-module with a basis $\{\overline{d(a_1)}, \overline{d(a_2)}, \ldots, \overline{d(a_{n-1})}, s(\chi)\}$ where for $a \in \mathbf{R}(A)$, we denote by $\overline{d(a)}$, the image of $d(a)$ in $\Omega_{\mathbf{R}(A)/\mathbf{R}}/\mathfrak{p}\Omega_{\mathbf{R}(A)/\mathbf{R}}$. As a consequence, $\wedge_{i=1}^{n-1}\overline{da_i} \wedge s(\chi) = \rho'$ is a generator for $(\mathbf{R}(K)/\mathfrak{p}\mathbf{R}(K))_f$.

Let $\rho \in K$ and $\tau \in L$ be generators of K_f and L_f respectively. Then, $T = \tau \otimes \rho$ is a generator for $\mathcal{E}_f$. As above, we can write $D_f \simeq A_f[T, T^{-1}]$ and we can consider the action of T on $\mathcal{P} = \Theta(\mathcal{M}_0, \omega_{\mathcal{M}_0})$ and $\mathcal{Q} = \Theta(\mathcal{M}_1, \omega_{\mathcal{M}_1})$. By (5.2),

$$sign(T(\mathcal{P})) = sgn_{(\tau,\rho)}(\mathcal{M}_0, \omega_{\mathcal{M}_0}), \quad sign(T(\mathcal{Q})) = sgn_{(\tau,\rho)}(\mathcal{M}_1, \omega_{\mathcal{M}_1}).$$

Since $\Pi : \Psi \to \gamma$ is bijective and $\gamma \subseteq X_f(\mathbf{R})$ we have $\Psi \subseteq Z_f(\mathbf{R})$. As T defines a continuous and semialgebraic function on $Z_f(\mathbf{R})$, and $\mathcal{P}, \mathcal{Q} \in \Psi$ (which is semialgebraically connected); $sign(T(\mathcal{P})) = sign(T(\mathcal{Q}))$. Hence, changing ρ to $-\rho$ if necessary, we may assume without loss of generality that,

$$sgn_{(\tau,\rho)}(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = 1 = sgn_{(\tau,\rho)}(\mathcal{M}_1, \omega_{\mathcal{M}_1}).$$

Now, as $\tau \in L$ is a generator of L_f by (4.6), there exists $a_n \in \mathbf{R}(A)$ and a surjection

$$\beta : \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-1} \twoheadrightarrow \mathcal{M}_0 \cap \mathcal{M}_1$$

such that

- $(a_1, a_2, \ldots, a_n) = \mathcal{M}_0 \cap \mathcal{M}_1$,
- $\beta(\tau) = a_1 + ha_n$ for some $h \in \mathbf{R}(A)$,
- $\beta(e_i) = a_i;\ 2 \leq i \leq n$ where $(e_2, \cdots, e_n)$ is a basis of $\mathbf{R}(A)^{n-1}$.

Moreover if $\bar{a_n}$ denotes the image of a_n in $\mathbf{R}(B)(= \mathbf{R}(A)/\mathfrak{p}\mathbf{R}(A)) \hookrightarrow \mathbf{R}(C)$, then

- $sign(d(\bar{a_n})/\chi) = -1$ at $\mathfrak{m}_0$ and $sign(d(\bar{a_n})/\chi) = 1$ at $\mathfrak{m}_1$

Note that if $d(\bar{a_n})/\chi = w \in \mathbf{R}(C) \subset \mathbf{R}(C)_f = \mathbf{R}(B)_f$, then $\wedge_{j=1}^n d\bar{a}_j = w\rho'$.

The surjection $\beta : \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-1} \twoheadrightarrow \mathcal{M}_0 \cap \mathcal{M}_1 (= (a_1, a_2, \ldots, a_n))$ gives rise to local orientations ω_0 and ω_1 of $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively as follows :

$$\omega_i : \frac{\mathbf{R}(L)}{\mathcal{M}_i \mathbf{R}(L)} \xrightarrow{\sim} \wedge^n (\frac{\mathcal{M}_i}{\mathcal{M}_i^2}) \quad , \quad \omega_i(\tau_i') = a_{i1} \wedge a_{i2} \wedge \ldots \wedge a_{in} \qquad i = 0, 1$$

where τ_i' denotes the image of τ in $\mathbf{R}(L)/\mathcal{M}_i\mathbf{R}(L)$ and a_{ij} denotes the image of a_j in $\mathcal{M}_i/\mathcal{M}_i^2$.

Recall that κ and τ are generators for $\mathbf{R}(L)^2$ and $\mathbf{R}(L)_f$ respectively. Therefore $\kappa = u(\tau \otimes \tau)$ for some $u \in \mathbf{R}(A)_f^*$. Since ρ is a generator of $\mathbf{R}(K)$ and ρ' is a generator of $(\mathbf{R}(K)/\mathfrak{p}\mathbf{R}(K))_f$, $\bar{\rho} = v\rho'$ for some $v \in \mathbf{R}(B)_f^*$ where $\bar{\rho}$ is the image of ρ in $(\mathbf{R}(K)/\mathfrak{p}\mathbf{R}(K))_f$.

Let u_i, v_i, w_i denote images of u, v, w in $\mathbf{R}(A)/\mathcal{M}_i$ respectively. Note that as γ is a semialgebraically connected subset of $\mathrm{Spec}(B_f)(\mathbf{R}) \subseteq X_f(\mathbf{R})$ and the points of $X(\mathbf{R})$ corresponding to $\mathcal{M}_0$ and $\mathcal{M}_1$ belong to γ,

$$sign(u_0)sign(u_1) = sign(v_0)sign(v_1) = 1.$$

On the other hand, by choice of w, $sign(w_0) = -1$ and $sign(w_1) = 1$. Now using the equalities $\kappa = u(\tau \otimes \tau), \bar{\rho} = v\rho'$ and $\wedge_{j=1}^n d\bar{a}_j = w\rho'$, we see that

$$sgn_{(\tau,\rho)}(\mathcal{M}_0, \omega_0) sgn_{(\tau,\rho)}(\mathcal{M}_1, \omega_1) = -1$$

But

$$sgn_{(\tau,\rho)}(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = 1 = sgn_{(\tau,\rho)}(\mathcal{M}_1, \omega_{\mathcal{M}_1}).$$

Without loss of generality we assume that

$$sgn_{(\tau,\rho)}(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = sgn_{(\tau,\rho)}(\mathcal{M}_0, \omega_0) = 1.$$

$$sgn_{(\tau,\rho)}(\mathcal{M}_1, \omega_{\mathcal{M}_1}) = -sgn_{(\tau,\rho)}(\mathcal{M}_1, \omega_1) = 1.$$

Hence,

$$(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = (\mathcal{M}_0, \omega_0), (\mathcal{M}_1, \omega_{\mathcal{M}_1}) = -(\mathcal{M}_1, \omega_1)$$

in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

Since orientations ω_0 and ω_1 on $\mathcal{M}_0$ and $\mathcal{M}_1$ are induced by the surjection $\beta : \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-1} \twoheadrightarrow \mathcal{M}_0 \cap \mathcal{M}_1$, we know that in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$,

$$(\mathcal{M}_0, \omega_0) + (\mathcal{M}_1, \omega_1) = 0.$$

Hence, in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$,

$$(\mathcal{M}_0, \omega_0) = -(\mathcal{M}_1, \omega_1) = (\mathcal{M}_1, -\omega_1)$$

and so

$$(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = (\mathcal{M}_1, \omega_{\mathcal{M}_1})$$

i.e. $(\mathcal{P}) = (\mathcal{Q})$. Thus, the lemma is proved. $\square$

We make a few comments about the choice of f in the previous lemma (5.4).

Lemma 5.5. *Let R be a regular domain of dimension n and let $\mathfrak{p}$ be a prime ideal of height $n-1$ such that the normalisation S of $R/\mathfrak{p}$ is a finite module over $R/\mathfrak{p}$. Let $\mathfrak{C}$ be an ideal of R containing $\mathfrak{p}$ such that $\mathfrak{C}/\mathfrak{p}$ is the conductor ideal of S over $R/\mathfrak{p}$. Let $a_1, a_2, \ldots, a_{n-1} \in R$ such that $(a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p} \cap I$ with $I \not\subset \mathfrak{p}$. Let L be a projective R-module of rank 1. Then there exists $f \in R \setminus \mathfrak{p}$ such that*

1. $f \in \mathfrak{C} \cap (\mathfrak{p} + I)$
2. $L_f \simeq R_f$.

Proof. Since $L_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$, there exists $g \in R \setminus \mathfrak{p}$ such that $L_g \simeq R_g$. Since $\mathfrak{p} \subsetneqq \mathfrak{C} \cap (\mathfrak{p} + I)$, there exists $h \in \mathfrak{C} \cap (\mathfrak{p} + I) \setminus \mathfrak{p}$. Now taking $f = gh$, we are through. $\square$

Remark 5.6. Let Ψ, γ be as in the lemma (5.4) and $\bar{\gamma}^{Zar} = \mathrm{Spec}(A/\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of A of height $n-1$. Since $\mathbf{R}(A)_{\mathfrak{p}\mathbf{R}(A)}$ is regular of dimension $n-1$, we can always find $a_1, a_2, \ldots, a_{n-1}$ such that $(a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p}\mathbf{R}(A) \cap I$

with $I \not\subset \mathfrak{p}\mathbf{R}(A)$. Then, (5.5) shows that there exists $f \in \mathbf{R}(A) \setminus \mathfrak{p}\mathbf{R}(A)$ satisfying conditions (1) and (2) in the hypothesis of (5.4). Note that if moreover $\gamma \subset X_f(\mathbf{R}) = (\mathrm{Spec}(A_f))(\mathbf{R})$ then for every maximal $\mathcal{M}$ of $\mathbf{R}(A)$ corresponding to a point of γ, $(B_1)_{\bar{\mathcal{M}}} = \mathbf{R}(B)_{\mathfrak{m}}$ is a discrete valuation ring where $B_1 = \mathbf{R}(A)/(a_1, a_2, \ldots, a_{n-1}), \mathbf{R}(B) = \mathbf{R}(A)/\mathfrak{p}\mathbf{R}(A), \overline{\mathcal{M}} = \mathcal{M}B_1$ and $\mathfrak{m} = \mathcal{M}\mathbf{R}(B)$. In particular every point of γ is a smooth point of $\mathrm{Spec}(A/\mathfrak{p})$ (recall $A/\mathfrak{p} = B$). Since an elementary path might contain non smooth points of the curve $\mathrm{Spec}(B)$, it is not always possible to have $f \in \mathbf{R}(A)$ satisfying condition (3) in the hypothesis of (5.4). However, $\mathrm{Spec}(\mathbf{R}(A)/f) \cap \gamma$ is a finite set, say, $\{Q_1, Q_2, \ldots, Q_t\}$. Since γ is totally ordered we can assume that $Q_i < Q_{i+1}; 1 \leq i \leq t-1$.

Let P and Q be the initial and end point of γ respectively. Then we have $P \leq Q_1 < Q_2 < \ldots < Q_t \leq Q$ and open intervals $]P, Q_1[,]Q_i, Q_{i+1}[, 1 \leq i \leq t-1$ and $]Q_t, Q[$ are contained in $X_f(\mathbf{R})$. Since $\Pi|_\Psi : \Psi \twoheadrightarrow \gamma$ is bijective and ordered preserving we have $\Pi_\Psi^{-1}(P) = \mathcal{P} \leq \Pi\_Psi^{-1}(Q_1) < \Pi\_\Psi^{-1}(Q_2) < \ldots < \Pi\_\Psi^{-1}(Q_t) \leq \mathcal{Q} = \Pi\_\Psi^{-1}(Q)$. Now let $\mathcal{P}', \mathcal{P}'' \in \Psi$ be such that $\Pi|_\Psi^{-1}(Q_i) < \mathcal{P}' < \mathcal{P}'' < \Pi_\Psi^{-1}(Q_{i+1})$, then the previous lemma says that $(\mathcal{P}') = (\mathcal{P}'')$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

The next lemma essentially says that $(\Pi|_\Psi^{-1}(Q_i)) = (\mathcal{P}') = (\mathcal{P}'') = (\Pi|_\Psi^{-1}(Q_{i+1}))$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

Lemma 5.7. *For every point $\mathcal{P}$ of $Z(\mathbf{R})$, there exists a semialgebraic neighbourhood $U_\mathcal{P}$ such that if $\mathcal{P}_1, \mathcal{P}_2 \in U_\mathcal{P}$, then $(\mathcal{P}_1) = (\mathcal{P}_2)$.*

Before proving this lemma, we state a standard lemma which we will require.

Lemma 5.8. *Let A be a smooth affine domain of dimension n over $\mathbf{R}$ and let $\mathcal{M}$ be a real maximal ideal of A. Let L be a rank 1 projective A-module. Assume that A is a surjective image of $\mathbf{R}^{[l]}$ where $\mathbf{R}^{[l]}$ denotes a polynomial algebra in l variables. Then there exists a set of variables $\{X_1, \cdots, X_l\}$ (i.e. $\mathbf{R}^{[l]} = \mathbf{R}[X_1, \cdots, X_l]$) and $f \notin \mathcal{M}$ such that A is a finite module over $\mathbf{R}[X_1, \cdots, X_n]$, $\Omega_{A_f/\mathbf{R}[X_1, \cdots, X_n]} = 0$ and $L_f \simeq A_f$.*

We now proceed to prove lemma (5.7).

Proof. Let $\Theta(\mathcal{M}, \omega_{\mathcal{M}}) = \mathcal{P}$. Let $P \in X(\mathbf{R})$ be the point corresponding to $\mathcal{M}$, i.e. $\Pi(\mathcal{P}) = P$. Since A is affine, we can assume that $\mathrm{Spec}(A)$ is a closed n-dimensional subvariety of the affine space $\mathbb{A}^l_{\mathbf{R}}$. Hence, $X(\mathbf{R})$ is a closed algebraic subset of $\mathbf{R}^l$. Then, by (5.8), there exists a suitable choice of a coordinate system of $\mathbf{R}^l$, such that the projection map $\pi : \mathbf{R}^l \to \mathbf{R}^n$ when restricted to $X(\mathbf{R})$ has finite fibers. Moreover, there exists $f \in A$ such that $f \notin \mathcal{M}$, $L_f \simeq A_f$ and $\Omega_{A_f/\mathbf{R}}$ is generated by $dX_1, \dots, dX_n$. Therefore, by the semialgebraic Inverse Function Theorem, there exists a semialgebraic Euclidean neighbourhood U of P contained in the Zariski neighbourhood $\mathrm{Spec}(A_f)$ such that the restriction of the projection map π to U is a Nash isomorphism onto an open ball $\mathcal{B}$ in $\mathbf{R}^n$ with center $\pi(P)$.

We fix an element $\tau \in L$ such that it generates L_f. Note that K_f is generated by $\rho = \wedge_{i=1}^n dX_i$. Note further that since $f \notin \mathcal{M}$, $\mathcal{P} \in Z_f(\mathbf{R})$. Let $D_f = A_f[T, T^{-1}]$, $T = \tau \otimes \rho$. Then,

$$Z_f(\mathbf{R}) = X_f(\mathbf{R}) \times \mathbf{R}^+ \bigsqcup X_f(\mathbf{R}) \times \mathbf{R}^-.$$

Without loss of generality we may assume that $sgn_{(\tau,\rho)}(\mathcal{P})$ is positive, i.e.

$$\mathcal{P} \in U \times \mathbf{R}^+ \subseteq X_f(\mathbf{R}) \times \mathbf{R}^+.$$

Let $U_{\mathcal{P}} = U \times \mathbf{R}^+$. By (2.15), $(\Theta^{-1}(\mathcal{M}, T-1)) = (\Theta^{-1}(\mathcal{M}, uT-1))$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ for $u > 0$. So we may assume that $\mathcal{P}$ corresponds to $(\mathcal{M}, T-1)$. Again by (2.15), to prove the proposition it is enough to prove that, $(\Theta^{-1}(\mathcal{M}, T-1)) = (\Theta^{-1}(\mathcal{M}', T-1))$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ for every $\mathcal{M}' \in U$. Let $\mathcal{P}'$ be the element of $Z(\mathbf{R})$ corresponding to $(\mathcal{M}', T-1)$.

Case 1. Suppose $\Pi(\mathcal{P}') = \Pi(\mathcal{P}) = P$. Then $\mathcal{P}' = \mathcal{P}$ and so $(\mathcal{P}) = (\mathcal{P}')$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

Case 2. Suppose $\Pi(\mathcal{P}') = P' \neq P$. Let W be a semialgebraic open subset of $\mathbf{R}^l$ such that $W \cap X(\mathbf{R}) = U$. Since $P \neq P', \pi(P) \neq \pi(P')$. Without loss of generality we assume that $\pi(P) = (0, \dots, 0)$ and $\pi(P') = (\delta_1, \dots, \delta_n)$. Moreover, without loss of generality, we assume that $\delta_n > 0$. The line $\mathcal{L}$ joining the two points $\pi(P) = (0, \dots, 0)$ and $\pi(P') = (\delta_1, \cdots, \delta_n)$ is given by $n-1$ equations :

$$H_i : X_i - \zeta_i X_n, \quad \zeta_i = \delta_i/\delta_n, \ 1 \leq i \leq n-1.$$

Let $\mathcal{L}_1$ be the segment of $\mathcal{L}$ contained in $\pi(U) = \mathcal{B}$. Then there exists an open interval $(a, b) \supset [0, \delta_n]$ and a Nash isomorphism from (a, b) to $\mathcal{L}_1$ given by :

$$t \mapsto (\zeta_1 t, \zeta_2 t, \cdots, \zeta_{n-1} t, t).$$

Composing the above function with $(\pi|_U)^{-1}$ we obtain a Nash embedding $F(t) = (f_1(t), \ldots, f_n(t), \ldots, f_l(t))$ from (a, b) to $(\pi|_U)^{-1}(\mathcal{L}_1) \subset U \subset \mathbf{R}^l$. It is easy to see that $f_i(t) = \zeta_i t$ for $i = 1, \ldots, n-1$, $f_n(t) = t$ and $f_j(t) = g_j(\zeta_1 t, \ldots, \zeta_{n-1} t, t)$ for $j = n+1, \ldots, l$. Note that $F(0) = P$ and $F(\delta_n) = P'$.

Let $\gamma = F([0, \delta_n])$. Then, $\gamma \subseteq U \subseteq \mathrm{Spec}(A_f)$. Moreover, since H_i vanishes at every point of γ for $1 \le i \le n-1$, $\gamma \subset (\mathrm{Spec}((A/(H_1, H_2, \ldots, H_{n-1})A)_f))(\mathbf{R})$. Note that $\Omega_{A_f/\mathbf{R}[X_1, X_2, \ldots, X_n]} = 0; \mathbf{R}[X_1, X_2, \ldots, X_n]/(H_1, H_2, \ldots, H_{n-1}) \simeq \mathbf{R}[T]$. Therefore, $(A/(H_1, H_2, \ldots, H_{n-1})A)_f$ is a regular ring. Hence onward we simply write $(H_1, H_2, \ldots, H_{n-1})$ for $(H_1, H_2, \ldots, H_{n-1})A$.

Consider

$$(H_1, H_2, \ldots, H_{n-1}) = (\cap_{i=1}^{r_1} \mathfrak{p}_i) \bigcap (\cap_{j=r_1+1}^{r_2} \mathfrak{p}_j) \bigcap (\cap_{k=1}^{r_3} \mathfrak{q}_k),$$

the primary decomposition of $(H_1, H_2, \ldots, H_{n-1})$ in A where

$$f \notin \cup_{i=1}^{r_1} \mathfrak{p}_i, f \in \cap_{j=r_1+1}^{r_2} \mathfrak{p}_j$$

and $\mathfrak{q}_k$ are primary but not prime ideals, which, as $(A/(H_1, H_2, H_{n-1}))_f$ is regular implies that

$$f^m \in \cap_{k=1}^{r_3} \mathfrak{q}_k, m \in \mathbb{N}.$$

Hence, $(H_1, H_2, \ldots, H_{n-1})_f = \cap_{i=1}^{r_1} (\mathfrak{p}_i)_f$. Then,

$$\mathrm{Spec}((A/(H_1, H_2, \ldots, H_{n-1}))_f) = \bigsqcup_{i=1}^{r_1} \mathrm{Spec}((A/(\mathfrak{p}_i))_f).$$

Since $\gamma \subset \mathrm{Spec}((A/(H_1, H_2, \ldots, H_{n-1}))_f) = \sqcup_{i=1}^{r_1} \mathrm{Spec}((A/\mathfrak{p}_i)_f)$, we have

$$\gamma = \bigsqcup_{i=1}^{r_1} \gamma \cap (\mathrm{Spec}((A/\mathfrak{p}_i)_f))(\mathbf{R}).$$

Since $\gamma \cap \mathrm{Spec}((A/\mathfrak{p}_i)_f)$ is a closed semialgebraic subset of γ and γ is semialgebraically connected, there exists i such that $\gamma = \gamma \cap \mathrm{Spec}(A/\mathfrak{p}_i)$ while the other intersections are empty. Without loss of generality let $i = 1$ and let us denote $\mathfrak{p}_1 = \mathfrak{p}$. Hence, $\gamma \subset \mathrm{Spec}(A/\mathfrak{p})$ and hence, $\bar{\gamma}^{Zar} = \mathrm{Spec}(A/\mathfrak{p})$. γ inherits a natural order from $[0, \delta_n]$ and every point of γ is a smooth point of $\mathrm{Spec}(A/(H_1, H_2, \ldots, H_{n-1}))$ as well as of $\mathrm{Spec}(A/\mathfrak{p})$. Hence, γ is an elementary path with $\Pi(\mathcal{P}) = P = F(0)$ as the initial point and $\Pi(\mathcal{P}') = P' = F(\delta_n)$ as the endpoint.

Consider the map $D_f \to A_f$ given by $T \mapsto 1$. This induces a section

$$s : \mathrm{Spec}(A_f) \to \mathrm{Spec}(D_f) \quad , \quad \mathcal{M} \mapsto (\mathcal{M}, T-1); \mathcal{M} \in Max(A_f).$$

Then let $\Psi = s(\gamma)$. Then clearly Ψ is an elementary path in $Z(\mathbf{R})$ with starting point $\mathcal{P}$ and endpoint $\mathcal{P}'$. Note that since it is a section, $\Pi(\Psi) = \gamma$ and the map $\Pi|_\Psi : \Psi \to \gamma$ is a bijection. Let

$$I = ((\cap_{i=2}^{r_1} \mathfrak{p}_i) \bigcap (\cap_{j=r_1+1}^{r_2} \mathfrak{p}_j) \bigcap (\cap_{k=1}^{r_3} \mathfrak{q}_k))\mathbf{R}(A).$$

Then

$$(H_1, H_2, \ldots, H_{n-1}) = \mathfrak{p}\mathbf{R}(A) \cap I.$$

Let $B = A/\mathfrak{p}$ and $B_1 = A/(H_1, H_2, \ldots, H_{n-1})$. Then since B_f and $(B_1)_f$ are regular, $f \in \sqrt{\mathfrak{C} \cap (\mathfrak{p}\mathbf{R}(A) + I)}$ where $\mathfrak{C}$ is the conductor ideal of $\mathbf{R}(A)$ with respect to its normalisation. By choice of f and γ, $L_f \simeq A_f \simeq K_f$ and $\mathrm{Spec}(\mathbf{R}(A)/f) \cap \gamma = \emptyset$. Therefore, we can apply (5.4) and hence we get that $(\mathcal{P}) = (\mathcal{P}')$. □

Proposition 5.9. *Let $\Psi \subset Z(\mathbf{R})$ be an elementary path and let $\Pi(\Psi) = \gamma$. Suppose $\Pi_\Psi : \Psi \to \gamma$ is bijective. Let $\mathcal{P}$ and $\mathcal{Q}$ be the initial and end points of Ψ. Then $(\mathcal{P}) = (\mathcal{Q})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.*

Proof. If Ψ is singleton then there is nothing to prove. So we assume that Ψ is nondegenerate. Then, γ is also a nondegenerate elementary path in $X(\mathbf{R})$.

Let $\bar{\Psi}^{Zar} = \mathrm{Spec}(D/\mathfrak{q})$ and $\bar{\gamma}^{Zar} = \mathrm{Spec}(A/\mathfrak{p})$. Then $\mathfrak{p} = A \cap \mathfrak{q}$ and $\mathfrak{p}$ is a prime ideal of height $n-1$ of A. Since $\mathbf{R}(A)$ is regular ; there exist $a_1, a_2, \ldots, a_{n-1} \in \mathbf{R}(A)$ such that $(a_1, a_2, \ldots, a_{n-1}) = \mathfrak{p}\mathbf{R}(A) \cap I$ with $I \not\subset \mathfrak{p}\mathbf{R}(A)$. Let B' be the normalisation of $\mathbf{R}(B)(= \mathbf{R}(A)/\mathfrak{p}\mathbf{R}(A))$ and let $\mathfrak{C}$ be an ideal of $\mathbf{R}(A)$ containing $\mathfrak{p}\mathbf{R}(A)$ such that $\mathfrak{C}/\mathfrak{p}\mathbf{R}(A) = \mathfrak{c}_{B'/\mathbf{R}(B)}$, the conductor ideal of B' over $\mathbf{R}(B)$.

Then, by (5.5), there exists $f \in \mathbf{R}(A) \setminus \mathfrak{p}$ such that $f \in \mathfrak{C} \cap (\mathfrak{p}\mathbf{R}(A) + I)$ and $\mathbf{R}(L)_f \simeq \mathbf{R}(A)_f$. Let $\Upsilon = \{T \in X(\mathbf{R}) | \mathfrak{p}\mathbf{R}(A) + (f) \subseteq \mathcal{M}_T\}$ where $\mathcal{M}_T$ denotes the maximal ideal of $\mathbf{R}(A)$ corresponding to T and $\Upsilon' = \Upsilon \cap \gamma$. Since $f \notin \mathfrak{p}\mathbf{R}(A)$ (an ideal of height $n-1$), Υ and hence Υ' are finite sets.

If $\Upsilon' = \emptyset$ then by lemma (5.4), $(\mathcal{P}) = (\mathcal{Q})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

So we assume $\Upsilon' \neq \emptyset$. Let $\Upsilon_1 = \Pi^{-1}(\Upsilon') \cap \Psi$. Then as $\Pi|_\Psi : \Psi \twoheadrightarrow \gamma$ is bijective, Υ_1 is a finite subset of Ψ, say $\Upsilon_1 = \{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_t\}$. Since Ψ is totally ordered, without loss of generality we assume that $\mathcal{Q}_i < \mathcal{Q}_{i+1}; 1 \leq i \leq t-1$. Let $\mathcal{P} = \mathcal{Q}_0$ and $\mathcal{Q} = \mathcal{Q}_{t+1}$. Then we have $\mathcal{Q}_0 \leq \mathcal{Q}_1 < \mathcal{Q}_2 < \ldots < \mathcal{Q}_t \leq \mathcal{Q}_{t+1}$.

Consider an interval $[\mathcal{Q}_i, \mathcal{Q}_{i+1}], 1 \leq i \leq t-1$. Then, by (5.7), there exists $U_{\mathcal{Q}_i} \subset Z(\mathbf{R})$ such that for any two points $\mathcal{S}, \mathcal{S}'$ in $U_{\mathcal{Q}_i}$, we have $(\mathcal{S}) = (\mathcal{S}')$. Note that by [11, Proposition 7.5] $\mathcal{Q}_i$ is contained in the closure of $]\mathcal{Q}_i, \mathcal{Q}_{i+1}[$. Therefore, $U_{\mathcal{Q}_i} \cap]\mathcal{Q}_i, \mathcal{Q}_{i+1}[\neq \emptyset$ (and hence is infinite). Choose $\mathcal{S}_{i,1} \in U_{\mathcal{Q}_i} \cap]\mathcal{Q}_i, \mathcal{Q}_{i+1}[$. Similarly, we can choose $\mathcal{S}_{i+1,0} \in U_{\mathcal{Q}_{i+1}} \cap]\mathcal{S}_{i,1}, \mathcal{Q}_{i+1}[$. Then, $[\mathcal{S}_{i,1}, \mathcal{S}_{i+1,0}]$ is a subinterval of $[\mathcal{Q}_i, \mathcal{Q}_{i+1}]$. Consider $\Psi_i = [\mathcal{S}_{i,1}, \mathcal{S}_{i+1,0}]$. Then note that $\bar{\Psi}_i^{Zar}$ is an infinite closed subset of the irreducible curve $\mathrm{Spec}(D/\mathfrak{q})$ and hence has to equal it. Hence, $\bar{\Psi}_i^{Zar} = \mathrm{Spec}(D/\mathfrak{q})$ and Ψ_i is actually an elementary path. Further, $\pi(\Psi_i) \cap \Upsilon = \emptyset$. Then, by (5.4), $(\mathcal{S}_{i,1}) = (\mathcal{S}_{i+1,0})$. Since $\mathcal{S}_{i,0}, \mathcal{S}_{i,1} \in U_{\mathcal{Q}_i}$, we also have $(\mathcal{S}_{i,0}) = (\mathcal{S}_{i,1})$. Hence, we get that $(\mathcal{S}_{1,1}) = (\mathcal{S}_{t,0})$. If $\mathcal{Q}_0 = \mathcal{Q}_1$, let $\mathcal{S}_{0,1} = \mathcal{S}_{1,1}$, else consider $[\mathcal{S}_{0,1}, \mathcal{S}_{1,0}]$. Similarly, if $\mathcal{Q}_t = \mathcal{Q}_{t+1}$, let $\mathcal{S}_{t,1} = \mathcal{S}_{t+1,1}$, else consider $[\mathcal{S}_{t,1}, \mathcal{S}_{t+1,0}]$. In all four cases, we get $(\mathcal{S}_{0,1}) = (\mathcal{S}_{t+1,0})$. But since $\mathcal{S}_{0,1}, \mathcal{Q}_0 \in U_\mathcal{P}$ and $\mathcal{S}_{t+1,0}, \mathcal{Q}_{t+1} \in U_\mathcal{Q}$, we have $(\mathcal{S}_{0,1}) = (\mathcal{Q}_0)$ and $(\mathcal{Q}_{t+1}) = (\mathcal{S}_{t+1,0})$. Hence, we get $(\mathcal{Q}_0) = (\mathcal{Q}_{t+1})$ i.e. $(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = (\mathcal{M}_1, \omega_{\mathcal{M}_1})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. ☐

Finally, we prove the main result of this section.

Theorem 5.10. *Let $A, L, K, \mathcal{E}, D, X(\mathbf{R}), Z(\mathbf{R}), Y, \Theta : Y \to Z(\mathbf{R})$ be as in the beginning of this section. Let $\mathcal{P} = \Theta((\mathcal{M}_0, \omega_{\mathcal{M}_0}))$ and $Q = \Theta((\mathcal{M}_1, \omega_{\mathcal{M}_1}))$ be two distinct points of $Z(\mathbf{R})$ lying in the same semialgebraically connected component of $Z(\mathbf{R})$. Then $(\mathcal{M}_0, \omega_{\mathcal{M}_0}) = (\mathcal{M}_1, \omega_{\mathcal{M}_1})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.*

Proof. Since $\mathcal{P}$ and $\mathcal{Q}$ lie in the same component of $Z(\mathbf{R})$, we can join $\mathcal{P}$ and $\mathcal{Q}$ by a semialgebraic path Ψ. Then by (2.12), this path breaks into finitely many non-degenerate elementary paths $\Psi_i, 1 \leq i \leq r$ such that $\Psi_i \cap \Psi_{i+1} = \{\mathcal{S}_i\}$ and

$\mathcal{S}_i$ is the starting point of Ψ_{i+1} and the endpoint of Ψ_i. Moreover, $\mathcal{P} = \mathcal{S}_0$ is the starting point of Ψ_1 and $\mathcal{Q} = \mathcal{S}_r$ is the end point of Ψ_r. Therefore it is enough to show that $(\mathcal{S}_i) = (\mathcal{S}_{i+1}), 0 \leq i \leq r-1$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

Hence we can assume without loss of generality that Ψ is a nondegenerate elementary path in $Z(\mathbf{R})$, with initial point $\mathcal{P}$ and end point $\mathcal{Q}$. If $\Pi(\Psi)$ (where $\Pi : Z(\mathbf{R}) \to X(\mathbf{R})$)) is singleton, then by (5.3) we are through. So we assume that $\Pi(\Psi)$ is not singleton (and hence infinite).

In this case, by [8, Theorem 3.1], there exists a sub-division $\mathcal{P} = \mathcal{P}_0 < \mathcal{P}_1 < \ldots < \mathcal{P}_t = \mathcal{Q}$ such that if $\Psi_j = [\mathcal{P}_j, \mathcal{P}_{j+1}]$, then $\Pi|_{\Psi_j} : [\mathcal{P}_j, \mathcal{P}_{j+1}] \to \Pi(\Psi_j)$ is order-preserving and bijective. Therefore, by (5.4) $(\mathcal{P}_i) = (\mathcal{P}_{i+1})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ for $0 \leq j \leq t-1$. Therefore $(\mathcal{P}) = (\mathcal{Q})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. $\square$

6 Structure theorem for $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$

In this section, we prove the structure theorem **Theorem B**.

We recall the setup once again. Let $X = \mathrm{Spec}(A)$ be a smooth affine variety over $\mathbf{R}$ of dimension $n \geq 2$. Assume further that the set $X(\mathbf{R})$ of real points is not empty, hence infinite. Let L be a projective A-module of rank 1. We denote $K_A = \wedge^n(\Omega_{A/\mathbf{R}})$ by K.

Let $\mathcal{E} = L \otimes_A K$. Let $D = \oplus_{-\infty < i < \infty} \mathcal{E}^i$. Let $Z = \mathrm{Spec}(D)$. Then there is a natural map $A \hookrightarrow D$ which gives rise to a natural surjection $Z \twoheadrightarrow X$ which induces a natural map $Z(\mathbf{R}) \twoheadrightarrow X(\mathbf{R})$ which we denote by Π. Looked at in the Euclidean topology, this gives an $\mathbf{R}^*$-bundle over $X(\mathbf{R})$.

Let
$$Y = \{(\mathcal{M}, \omega_{\mathcal{M}}) | \mathcal{M} \in X(\mathbf{R}), \quad \omega_{\mathcal{M}} : \frac{L}{\mathcal{M}L} \xrightarrow{\sim} \wedge^n(\frac{\mathcal{M}}{\mathcal{M}^2})\}.$$
Then there is a natural bijection $\Theta : Y \xrightarrow{\sim} Z(\mathbf{R})$. Recall that $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ is a quotient of the free abelian group on the set Y.

Let $C_1, C_2, \ldots, C_r, C_{r+1}, \ldots, C_t$ be the closed and bounded components of $X(\mathbf{R})$.

Note that $L, K, \mathcal{E} = L \otimes_A K$ correspond to semialgebraic line bundles on $X(\mathbf{R})$. Let $L_i, K_i, \mathcal{E}_i$ be the restrictions of these line bundles to C_i. Then $\Pi^{-1}(C_i)$ is the complement of the zero section of $\mathcal{E}_i$. Note that L_i is isomorphic to K_i as a semialgebraic line bundle (denoted by $L_i \simeq K_i$) if and only if $\mathcal{E}_i$ is a semialgebraically trivial line bundle over C_i.

Now suppose that $L_i \simeq K_i$ for $1 \leq i \leq r$ and $L_i \not\simeq K_i$ for $r+1 \leq i \leq t$.

Lemma 6.1. *$2(\mathcal{M}, \omega_{\mathcal{M}}) = 0$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ where the point corresponding to $\mathcal{M}$ lies in $C_i, r+1 \leq i \leq t$.*

Proof. Since $\Pi : Z(\mathbf{R}) \to X(\mathbf{R})$ is a continuous semialgebraic map, every component of $Z(\mathbf{R})$ is contained in $\Pi^{-1}(C)$ for some component C of $X(\mathbf{R})$. In particular, if $\Pi^{-1}(C)$ is semialgebraically connected then $\Pi^{-1}(C)$ is a semialgebraically connected component of $Z(\mathbf{R})$. Note that by (2.5), if C is a closed and bounded component of $X(\mathbf{R})$ then $\Pi^{-1}(C)$ has two components if and only if $\mathcal{E}|_C$ is trivial, otherwise $\Pi^{-1}(C)$ is semialgebraically connected.

Note that since $\mathcal{E}_i$ is not trivial for $r+1 \leq i \leq t$, by (2.5), $\Pi^{-1}(C_i)$ is semialgebraically connected and hence is a component of $Z(\mathbf{R})$. Now if $\mathcal{M}$ is a maximal ideal of $\mathbf{R}(A)$ such that the point corresponding to it lies in C_i then $\Theta((\mathcal{M}, \omega_{\mathcal{M}}))$ and $\Theta((\mathcal{M}, -\omega_{\mathcal{M}})) \in \Pi^{-1}(C_i)$. Since $\Pi^{-1}(C_i)$ is semialgebraically connected for $r+1 \leq i \leq t$, by (5.10), $(\mathcal{M}, \omega_{\mathcal{M}}) = (\mathcal{M}, -\omega_{\mathcal{M}})$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. Therefore, by (2.16), $2(\mathcal{M}, \omega_{\mathcal{M}}) = 0$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. $\square$

Lemma 6.2. *Let $\mathcal{M}$ be a maximal ideal of $\mathbf{R}(A)$ corresponding to a point T' in C, where C is an unbounded component of $X(\mathbf{R})$. Let $\omega_{\mathcal{M}}$ be a local L-orientation of $\mathcal{M}$. Then $(\mathcal{M}, \omega_{\mathcal{M}}) = 0$ in $E(\mathbf{R}(A), \mathbf{R}(L)$.*

Proof. Let $\bar{X}$ be the smooth projective completion of $X = \mathrm{Spec}(A)$. Then there exists an affine open subset $X_1 = \mathrm{Spec}(A_1)$ of $\widetilde{X}$ such that $X_1(\mathbf{R}) = \bar{X}(\mathbf{R})$. Then if $X' = X \cap X_1$, we have $X'(\mathbf{R}) = X(\mathbf{R})$. Let A' be the coordinate ring of X'. Since $X \cap X_1$ is an affine open subset of X_1 and $\mathrm{Pic}(\mathbf{R}(A_1))$ is a 2-torsion group, $\mathbf{R}(A')$ is a localization of $\mathbf{R}(A_1)$. Now since $X \cap X_1$ is an open subset of X and $X'(\mathbf{R}) = X(\mathbf{R})$, we have $\mathbf{R}(A') = \mathbf{R}(A)$. Let L_1 be a rank 1 projective over A_1 such that L_1 and L define the same projective module over $\mathbf{R}(A)$. Note that since $\widetilde{X}$ is projective, $X_1(\mathbf{R}) = \widetilde{X}(\mathbf{R})$ is closed and bounded.

Since $X'(\mathbf{R}) = X(\mathbf{R})$, we can regard C as a semialgebraically connected subset of $X_1(\mathbf{R})$. Therefore there exists a component $\tilde{C}$ of $X_1(\mathbf{R})$ such that $C \subset \tilde{C}$. Since $\tilde{C}$ is closed and bounded and C is not closed and bounded, there exists $T \in \tilde{C}$

such that $T \notin C$. Note that $\tilde{C} \not\subset X(\mathbf{R})$ (otherwise $\tilde{C}$ being semialgebraically connected, $\tilde{C} \subset C$). Therefore we can assume that $T \notin X(\mathbf{R})$.

Let $\mathcal{M}_T$ denote the corresponding maximal ideal of $\mathbf{R}(A_1)$. Since $T \notin X(\mathbf{R})$, we have $\mathcal{M}_T\mathbf{R}(A) = \mathbf{R}(A)$. Let $\omega_{\mathcal{M}_T}$ be a local L_1-orientation of $\mathcal{M}_T$. Since $T, T' \in \tilde{C}$, by (5.10), either $(\mathcal{M}_T, \omega_{\mathcal{M}_T}) = (\mathcal{M}, \omega_{\mathcal{M}})$ or $(\mathcal{M}_T, \omega_{\mathcal{M}_T}) = (\mathcal{M}, -\omega_{\mathcal{M}})$ in $\mathrm{E}(\mathbf{R}(A_1), \mathbf{R}(L_1))$. Since $\mathbf{R}(A) = \mathbf{R}(A')$ is a localization of $\mathbf{R}(A_1)$, there exists a (surjective) group homomorphism from $\mathrm{E}(\mathbf{R}(A_1), \mathbf{R}(L_1))$ to $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. Since under this group homomorphism, $(\mathcal{M}_T, \omega_{\mathcal{M}_T}) \mapsto 0$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$, $(\mathcal{M}, \omega_{\mathcal{M}}) = 0$ in $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. □

Let $T_i \in C_i,\ 1 \le i \le t$ and let $\mathcal{M}_i$ be the maximal ideal of $\mathbf{R}(A)$ corresponding to T_i. Let ω_i be a local L-orientation of $\mathcal{M}_i$ for $1 \le i \le t$. Let F be a free abelian group with a basis $(e_1, \cdots, e_t)$. Let $\Delta : F \to \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ be a group homomorphism defined by $\Delta(e_i) = (\mathcal{M}_i, \omega_i);\ 1 \le i \le t$.

Proposition 6.3. *Δ is surjective. As a consequence $\mathrm{E}_0(\mathbf{R}(A), \mathbf{R}(L))$ is a vector space of rank $\le t$ over $\mathbb{Z}/(2)$.*

Proof. Since $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ is generated by Y, it is enough to show that all elements of Y are in the image of Δ. Let $(\mathcal{M}, \omega_{\mathcal{M}}) \in Y$. Suppose the point corresponding to $\mathcal{M}$ does not lie in C_i for any $i, 1 \le i \le t$. Then, by (6.2), $(\mathcal{M}, \omega_{\mathcal{M}}) = 0$ and hence, it trivially lies in the image of Δ. If the point corresponding to $\mathcal{M}$ lies in C_i for some $i, 1 \le i \le t$, then, using (5.10), $(\mathcal{M}, \omega_{\mathcal{M}}) = (\mathcal{M}_i, \omega_i)$ or $(\mathcal{M}, \omega_{\mathcal{M}}) = (\mathcal{M}_i, -\omega_i) = -(\mathcal{M}_i, \omega_i)$. Hence, $(\mathcal{M}, \omega_{\mathcal{M}}) = \Delta(e_i)$ or $(\mathcal{M}, \omega_{\mathcal{M}}) = -\Delta(e_i)$. Therefore, Δ is surjective.

By (2.16), $\mathrm{E}_0(\mathbf{R}(A), \mathbf{R}(L))$ is a vector space over $\mathbb{Z}/(2)$ and $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ maps surjectively onto it. Hence, it is a vector space of rank $\le t$ over $\mathbb{Z}/(2)$. □

Remark 6.4. By (6.1), the map Δ induces the surjection :

$$\oplus_{i=1}^{r} \mathbb{Z} e_i \bigoplus \oplus_{i=r+1}^{t} (\mathbb{Z}/2) e_i \overset{\bar{\Delta}}{\twoheadrightarrow} \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L)).$$

We will prove that $\bar{\Delta}$ is an isomorphism.

Recall that there is a bijection $\Theta : Y \xrightarrow{\sim} Z(\mathbf{R})$. Note that for $1 \leq i \leq r$, there is a section $C_i \to \Pi^{-1}(C_i)$ which induces the following commutative diagram :

$$\begin{array}{ccccc} \mathbf{R}^* & \xleftarrow{p_2} & C_i \times \mathbf{R}^* & \xleftarrow[\sim]{s_i} & Z(\mathbf{R})|_{C_i} \\ & & & \searrow^{p_1} & \downarrow \Pi \\ & & & & C_i \end{array}$$

Then, we have a map $Z(\mathbf{R}) \to \{-1, 0, 1\}$ sending

$$\mathcal{P} \mapsto \begin{cases} 0 & \text{if } \Pi(\mathcal{P}) \notin C_i \\ -1 & \text{if } \Pi(\mathcal{P}) \in C_i \text{ and } p_2(s_i(\mathcal{P})) < 0 \\ 1 & \text{if } \Pi(\mathcal{P}) \in C_i \text{ and } p_2(s_i(\mathcal{P})) > 0 \end{cases}$$

Consider the composite map $sign_i : Y \to \{-1, 0, 1\}$ and consider the induced map on the free abelian group on Y to $\mathbb{Z}$. Note that if $(\mathcal{M}, \omega_0)$ and $(\mathcal{M}, \omega_1)$ are two local orientations of $\mathcal{M}$ (where $\mathcal{M}$ corresponds to a point in C_i), then there exists $\lambda \in \mathbf{R}^*$ such that $\omega_0 = \lambda\omega_1$. Then, $p_2(s_i(\Theta(\mathcal{M}, \omega_0))) = \lambda p_2(s_i(\Theta(\mathcal{M}, \omega_1)))$ and hence, $sign_i((\mathcal{M}, \omega_0)) = sign(\lambda) sign_i((\mathcal{M}, \omega_1))$.

Recall that $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$ is a quotient of the free abelian group on Y. The next lemma shows that $sign_i$ in fact factors through $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

Lemma 6.5. *Let I be a finite intersection of maximal ideals of $\mathbf{R}(A)$ and let $\beta : \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-1} \twoheadrightarrow I$ be a surjection. Let $1 \leq i \leq r$ and let $\mathcal{M}'_1, \ldots, \mathcal{M}'_l$ be all the maximal ideals of $\mathbf{R}(A)$ such that :*

1. *$I \subseteq \mathcal{M}'_j, 1 \leq j \leq l$.*
2. *the point corresponding to $\mathcal{M}'_j$ is contained in C_i, $1 \leq j \leq l$.*

Let ω_j be the local $\mathbf{R}(L)$- orientation of $\mathcal{M}'_j$ induced by β. Then

$$\sum_{j=1}^{l} sign_i((\mathcal{M}'_j, \omega_j)) = 0.$$

Proof. Let $\{f_2, f_3, \cdots, f_n\}$ be a basis of $\mathbf{R}(A)^{n-1}$ and let $\beta(f_k) = a_k,\ 2 \leq k \leq n$. We can assume that if $J = (\beta(\mathbf{R}(L)), a_2, \ldots, a_{n-1})$ then J is a prime ideal of height $n-1$ and $\mathrm{Spec}(A/J)$ is a smooth irreducible curve [14, Theorems 1.3 and

1.4]. Let "tilde" denote reduction modulo the ideal J. Then we have the following exact sequence

$$0 \longrightarrow \frac{J}{J^2} \overset{\tilde{a} \mapsto \widetilde{d(a)}}{\longrightarrow} \frac{\Omega_{\mathbf{R}(A)/\mathbf{R}}}{J\Omega_{\mathbf{R}(A)/\mathbf{R}}} \longrightarrow \Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}} \longrightarrow 0.$$

Since $\mathbf{R}(\tilde{A})$ is smooth, we can apply (2.7) and hence, $\Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}$ is a free $\mathbf{R}(\tilde{A})$-module of rank one. Let θ be a generator of $\Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}$ and let v be its preimage in $\Omega_{\mathbf{R}(A)/\mathbf{R}}/J(\Omega_{\mathbf{R}(A)/\mathbf{R}})$. Let $F = \mathbf{R}(L) \oplus \mathbf{R}(A)^{n-2}$. Then, since $\beta(F) = J$ we get an isomorphism $\widetilde{\beta} : F/JF \simeq J/J^2$. Hence $\Omega_{\mathbf{R}(A)/\mathbf{R}}/J\Omega_{\mathbf{R}(A)/\mathbf{R}} \simeq F/JF \oplus \Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}$. Since $\wedge^{n-1}(F) \simeq \mathbf{R}(L)$ and $\wedge^n(\Omega_{\mathbf{R}(A)/\mathbf{R}}) = \mathbf{R}(K)$, we get that

$$\frac{\mathbf{R}(L)}{J\mathbf{R}(L)} \overset{\sim}{\to} \wedge^n(\frac{F}{JF} \oplus \Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}) \overset{\sim}{\to} \wedge^n(\frac{J}{J^2} \oplus \Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}) \overset{\sim}{\to} \wedge^n(\frac{\Omega_{\mathbf{R}(A)/\mathbf{R}}}{J\Omega_{\mathbf{R}(A)/\mathbf{R}}}) = \frac{\mathbf{R}(K)}{J\mathbf{R}(K)}$$

$$\tilde{l} \mapsto (\tilde{l} \wedge (\wedge_{i=2}^{n-1} \tilde{f}_i)) \otimes \theta \mapsto (\widetilde{\beta(l)} \wedge (\wedge_{i=2}^{n-1} \tilde{a}_i)) \otimes \theta \mapsto \widetilde{d(\beta(l))} \wedge (\wedge_{i=2}^{n-1} \widetilde{d a_i}) \wedge v.$$

This induces a natural map

$$\frac{\mathbf{R}(A)}{J\mathbf{R}(A)} \overset{\sim}{\to} \frac{\mathbf{R}(L)}{J\mathbf{R}(L)} \otimes \frac{\mathbf{R}(L)}{J\mathbf{R}(L)} \overset{\sim}{\to} \frac{\mathbf{R}(\mathcal{E})}{J\mathbf{R}(\mathcal{E})}.$$

Let $\chi \in \mathbf{R}(\mathcal{E})$ be such that $1 \mapsto \tilde{\kappa} \mapsto \tilde{\chi}$ under the above map. This induces a map $\mathbf{R}(D)/J\mathbf{R}(D) \twoheadrightarrow \mathbf{R}(A)/J\mathbf{R}(A)$ mapping $\tilde{\chi} \mapsto 1$. This gives another section of the line bundle restricted to $(V(J))(\mathbf{R}) \subset X(\mathbf{R})$. Let "bar" denote reduction modulo $\mathcal{M}'_j$. Let $\bar{\omega}_j$ be the orientation sending $\bar{l} \mapsto \bar{\beta(l)} \wedge (\wedge_{i=2}^{n-1} \bar{a}_i) \wedge d^{-1}_{\mathcal{M}'_j}(\bar{v})$ for each j. Then, $\Theta(\mathcal{M}'_j, \bar{\omega}_j) = (\mathcal{M}'_j, \chi - 1)$ for each j.

Note that $\tilde{I} = (\tilde{a_n})$. Since θ is a generator of $\Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}$, we have $d(\tilde{a_n}) = u\theta$ for some $u \in \mathbf{R}(\tilde{A})$. Note that $d(\tilde{a_n})$ is non-zero in $\Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}/\mathcal{M}'_j\Omega_{\mathbf{R}(\tilde{A})/\mathbf{R}}$ and hence is a generator. Hence, u is a unit modulo $\mathcal{M}'_j$, i.e $u_j = u(\mathcal{M}'_j) \in \mathbf{R}^*$. Therefore,

$$\overline{d(\beta(l))} \wedge (\wedge_{i=2}^{n-1} d\bar{a}_i) \wedge \bar{v} = u_j^{-1} \overline{d(\beta(l))} \wedge (\wedge_{i=2}^{n} d\bar{a}_i).$$

Hence,

$$\overline{\beta(l)} \wedge (\wedge_{i=2}^{n-1} \bar{a}_i) \wedge d^{-1}_{\mathcal{M}'_j}(\bar{v}) = u_j^{-1} \overline{\beta(l)} \wedge (\wedge_{i=2}^{n} \bar{a}_i).$$

Hence, $\bar{\omega}_j = u_j^{-1}\omega_j$ for every j. Hence, $sign_i(\mathcal{M}'_j, \bar{\omega}_j) = sign(u_j) sign_i(\mathcal{M}'_j, \omega_j)$ and so, $sign(u_j) sign_i(\mathcal{M}'_j, \bar{\omega}_j) = sign_i(\mathcal{M}'_j, \omega_j)$.

Since $(V(J))(\mathbf{R}) \subset X(\mathbf{R})$, any component of $(V(J))(\mathbf{R})$ is disjoint from C_i or completely contained in it. Hence, $C_i \cap (V(J))(\mathbf{R}) = \sqcup_k W_{ik}$ where W_{ik} are components of $(V(J))(\mathbf{R})$. Then, we have

$$\sum_{j=1}^{l} sign_i((\mathcal{M}'_j, \omega_j)) = \sum_k \sum_{j:\mathcal{M}'_j \in W_{ik}} sign_i((\mathcal{M}'_j, \omega_j)).$$

Hence, it is enough to show that for each W_{ik}, $\sum_{j:\mathcal{M}'_j \in W_{ik}} sign_i((\mathcal{M}'_j, \omega_j)) = 0$. We note that W_{ik} are closed subsets of C_i and hence are closed and bounded components of $(V(J))(\mathbf{R})$ and hence components of its "completion".

Fix $W_{ik} = W$. Since W is a closed and bounded component of $(V(J))(\mathbf{R})$ and hence a component of its "completion", by [10, Theorem 3.3], we get that

$$\sum_{j:\mathcal{M}'_j \in W} sign(\frac{d(\widetilde{a_n})}{\theta}(\mathcal{M}'_j)) = \sum_{j:\mathcal{M}'_j \in W} sign(u_j) = 0.$$

Now,

$$\sum_{j:\mathcal{M}'_j \in W} sign_i((\mathcal{M}'_j, \omega_j)) = \sum_{j:\mathcal{M}'_j \in W} sign(u_j) sign_i((\mathcal{M}'_j, \bar{\omega}_j)).$$

By definition, $sign_i((\mathcal{M}'_j, \bar{\omega}_j)) = sign(p_2(s_i(\Theta((\mathcal{M}'_j, \bar{\omega}_j)))))$. Since $\Theta((\mathcal{M}'_j, \bar{\omega}_j)) = (\mathcal{M}, \chi - 1)$, the section induced by χ on $(V(J))(\mathbf{R})$ is given by the map $\mathcal{M} \mapsto \Theta((\mathcal{M}'_j, \omega)) = (\mathcal{M}, \chi - 1)$. Hence, $p_2(s_i(\Theta((\mathcal{M}'_j, \omega))))$ is a continuous, semialgebraic map on W which is semialgebraically connected. Hence, $p_2(s_i(\mathcal{M}, \chi - 1))$ has the same sign for all $\mathcal{M} \in W$ and so,

$$\sum_{j:\mathcal{M}'_j \in W} sign(u_j) sign_i((\mathcal{M}'_j, \omega)) = \pm \sum_{j:\mathcal{M}'_j \in W} sign(u_j) = 0.$$

Hence, the lemma is proved.

□

Thus, the map $sign_i$ factors through $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$. Without loss of generality we can assume that for the chosen generators $(\mathcal{M}_i, \omega_i)$ of $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$, $sign_i((\mathcal{M}_i, \omega_i)) = 1$ for $1 \leq i \leq r$. Using this, we obtain the structure theorem for $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$.

Theorem 6.6.

$$\oplus_{i=1}^{r}\mathbb{Z}e_i \bigoplus \oplus_{i=r+1}^{t}(\mathbb{Z}/2)e_i \overset{\bar{\Delta}}{\twoheadrightarrow} \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L))$$

is an isomorphism.

Proof. We recall that there is a natural map from $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L)) \twoheadrightarrow \mathrm{CH}_0(X)/G$ where $G = \pi_*(\mathrm{CH}_0(\bar{X}))$ and from (3.1), $\mathrm{CH}_0(X)/G \simeq (\mathbb{Z}/(2))^t$ where every point of any component is a generator. Hence, for each $i : r+1 \leq i \leq t$, there is a natural surjection $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L)) \twoheadrightarrow \mathbb{Z}/(2)$ obtained by first taking the surjection $\mathrm{E}(\mathbf{R}(A), \mathbf{R}(L)) \twoheadrightarrow \mathrm{CH}_0(X)/G(\simeq (\mathbb{Z}/(2))^t)$ followed by the projection to the i^{th} factor of $(\mathbb{Z}/(2))^t$. we denote it by $sign_i$ (slightly abusing notation). Then, putting together these maps $sign_i, 1 \leq i \leq t$, we get a map $\Delta' : \mathrm{E}(\mathbf{R}(A), \mathbf{R}(L)) \to \oplus_{i=1}^{r}\mathbb{Z}e_i \bigoplus \oplus_{i=r+1}^{t}(\mathbb{Z}/2)e_i$. sending $(\mathcal{M}, \omega_{\mathcal{M}}) \mapsto \sum_{i=1}^{t} sign_i((\mathcal{M}, \omega_{\mathcal{M}}))e_i$. Since $\Delta' \circ \bar{\Delta} = id$, we get that $\bar{\Delta}$ is an isomorphism. This finishes the proof. □

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