EULER CLASS GROUPS AND 2-TORSION ELEMENTS

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Abstract. We exhibit an example of a smooth affine threefold A over a field of characteristic 0 for which there exists non trivial 2-torsion elements in the Euler class group $E(A)$ vanishing in the weak Euler class group $E_0(A)$. This gives a positive answer to a question of the first author and Raja Sridharan.

Introduction

Let $X = \mathrm{Spec}(A)$ be a smooth affine scheme of dimension d over a field k and let E be a vector bundle of rank r over X. If E decomposes as $E \simeq E' \oplus \mathcal{O}_X$, then the top Chern class $c_r(E)$ vanishes in $CH^r(X)$. When $r = d = 3$ and k is algebraically closed, N. Mohan Kumar and M. Pavaman Murthy proved in [13] that the converse statement holds, i.e. $E \simeq E' \oplus \mathcal{O}_X$ if and only if $c_3(E) = 0$. This result was further extended by M. Pavaman Murthy to the case $d = r \in \mathbb{N}$ and k algebraically closed. The well-known example of the tangent bundle to the real algebraic 2-sphere shows that such a result is in general not true over an arbitrary field.

To understand when a vector bundle of rank $r = d$ splits off a free factor of rank one, the first author and Raja Sridharan introduced the Euler class groups $E(A, L)$ of A with coefficients in a line bundle L (following an original idea of M. V. Nori), and associated to any projective module P its Euler class $e(P, \chi) \in E(A, L)$, where $\chi : \det P \xrightarrow{\sim} L$ is any isomorphism. Then the Euler class $e(P, \chi)$ vanishes if and only if P splits off a free factor of rank one ([5, Corollary 4.4]).

Around the same period, J. Barge and F. Morel associated to any smooth scheme X, any $r \in \mathbb{N}$ and any line bundle L over X modified versions of Chow groups, now called Chow-Witt groups, denoted by $\widetilde{CH}^r(X, L)$ ([2] and [8]). If E is a vector bundle of rank r then it has an associated Euler class $\tilde{c}(E) \in \widetilde{CH}^r(X, \det E)$ which vanishes when $E \simeq E' \oplus \mathcal{O}_X$. If X is of dimension d and E is of rank d, then the analogue of the result in the setting of Euler class groups also holds, i.e. $\tilde{c}(E) = 0$ if and only if $E \simeq E' \oplus \mathcal{O}_X$ ([8, Corollary 15.3.12], [7, Theorem 38] and [16, Theorem 6]).

The two constructions are deeply related. By construction, there is a surjective homomorphism $E(A, L) \to \widetilde{CH}^d(X, L)$ and the question to know whether it is also injective is still open (for positive results in this direction, see Section 2).

In any case, these two theories provide a very good framework to understand when a vector bundle of top rank splits off a free factor of rank one. However, recall that the starting point of the story is the example of the tangent bundle T to the real algebraic 2-sphere. This vector bundle is stably free and indecomposable. If X

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is smooth of *odd* dimension d over a field k, then any stably free module E of rank d decomposes as $E = E' \oplus \mathcal{O}_X$, and thus an example in the nature of T would be impossible in odd dimensions. Indeed, the following question is still open:

Question 1. *Let X be a smooth affine d-fold over a field k, where d is odd. Let E be a vector bundle of rank d whose top Chern class $c_d(E)$ vanishes in $CH^d(X)$. Do we have $E \simeq E' \oplus \mathcal{O}_X$?*

This question has a positive answer when $d = 1$ (obvious) and for any odd d when $k = \mathbb{R}$ by [3, Theorem 4.30].

In the context of the Euler class groups, there is an analogue of the Chow group $CH^d(X)$ called the weak Euler class group and denoted by $E_0(A)$ (it is independent of L). One can associate to any projective module P of rank d a weak Euler class $e(P) \in E_0(A)$. There is a canonical surjective homomorphism $E_0(A) \to CH^d(X)$ mapping $e(P)$ to $c_d(P)$ and this homomorphism is an isomorphism if $k = \mathbb{R}$ or k is algebraically closed. In general, this homomorphism is conjecturally an isomorphism. In any case, there is a commutative diagram

$$\begin{array}{ccc} E(A,L) & \longrightarrow & E_0(A) \\ \downarrow & & \downarrow \\ \widetilde{CH}^d(X,L) & \longrightarrow & CH^d(X) \end{array}$$

showing that the weak Euler class group is to the Euler class group what the Chow group is to the Chow-Witt group.

In this setup, Raja Sridharan and the first author observe that the Euler class $e(P,\chi) \in E(A, \det P)$ is 2-torsion if the weak Euler class $e(P)$ vanishes in $E_0(A)$. This lead them to ask the following question ([5, Question 7.11]):

Question 2. *Can the kernel of the the canonical homomorphism $E(A,L) \to E_0(A)$ have non-trivial 2-torsion?*

A negative answer to this question would clearly imply that the weak Euler class $e(P)$ vanishes if and only if $P \simeq P' \oplus A$, thus solving the Euler class theory analogue of Question 1. It is known that Question 2 has a negative answer when $k = \mathbb{R}$.

In this note, we give an example to show that Question 2 has an affirmative answer. More precisely, we construct a smooth affine algebra A over the field $k = \mathbb{Q}(i)(t_1, t_2, \ldots, t_n)$ (where $n \geq 3$) such that the kernel of the homomorphism $E(A) \to E_0(A)$ is an $\mathbb{F}_2$-vector space of dimension $\geq n-2$ (Theorem 4.1 in the text). Thus the hope to show that Question 2 had a negative answer to solve Question 1 was too much too ask. However, in the algebra A we constructed, any rank 3 projective module P with trivial determinant is of the form $P = Q \oplus A$ and thus Question 1 still remains tantalizingly open.

The paper is organized as follows: In Sections 1 and 2 we briefly recall the definitions of the Euler class groups, the weak Euler class groups and the Chow-Witt groups. After these preliminaries, we construct our example in Section 3 using the full power of the functorial properties of the Chow-Witt groups. This allows to show in Section 4 that Question 2 has a positive answer. In the last section, we prove that any projective module of rank 3 over the algebra A constructed in Section 3 is of the form $P = Q \oplus A$.

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Conventions. The fields considered are of characteristic different from 2. If X is a scheme over a field k and $x_p \in X^{(p)}$, we denote by $\mathfrak{m}_p$ the maximal ideal in $\mathcal{O}_{X,x_p}$ and by $k(x_p)$ its residue field. Finally ω_{x_p} will denote the $k(x_p)$-vector space $\mathrm{Ext}^p_{\mathcal{O}_{X,x_p}}(k(x_p), \mathcal{O}_{X,x_p})$ (which is one-dimensional if X is regular at x_p).

1. Euler class groups

Let $X = \mathrm{Spec}(A)$ be a smooth, affine k-variety of dimension d and P be a projective A-module of rank r. The top Chern class $c_r(P) \in CH^r(X)$ provides a natural obstruction to the existence of a free direct summand of P. Unfortunately, very often these are not exact obstructions, i.e. there are projective modules which do not split off free summands of rank 1 even though the top Chern class is 0. Euler class groups originated in order to obtain an exact obstruction class for such a splitting to exist when $d = r$. We define them below.

1.1. **Definition of** $\mathrm{E}(A, L)$ **and** $\mathrm{E}_0(A, L)$**.** Let A be a smooth affine k-domain of dimension $d \geq 2$ and let L be a projective A-module of rank 1. Let $\mathcal{M}$ be a maximal ideal of A of height d. Then, $\mathcal{M}/\mathcal{M}^2$ is generated by d elements. An isomorphism $\omega_{\mathcal{M}} : L/\mathcal{M}L \xrightarrow{\sim} \wedge^d(\mathcal{M}/\mathcal{M}^2)$ is called a *local L-orientation* of $\mathcal{M}$. Let G be the free abelian group on the set of pairs $(\mathcal{M}, \omega_{\mathcal{M}})$ where $\mathcal{M}$ is a maximal ideal of height d and $\omega_{\mathcal{M}}$ is a local L-orientation of $\mathcal{M}$.

Let $J = \cap_{i=1}^n \mathcal{M}_i$ be an intersection of finitely many maximal ideals of height d. Then, J/J^2 is generated by d elements. An isomorphism $L/JL \xrightarrow{\sim} \wedge^d(J/J^2)$ is called a *local L-orientation* of J. Localizing, we see that any local L-orientation of J gives rise to local $\mathcal{M}_i$-orientations $\omega_{\mathcal{M}_i}$ of $\mathcal{M}_i$ for $i = 1, 2, \ldots, n$. We denote the element $\sum_{i=1}^n (\mathcal{M}_i, \omega_{\mathcal{M}_i})$ in G as (J, ω_J).

A local L-orientation $\omega : L/JL \twoheadrightarrow \wedge^d(J/J^2)$ is called a *global L-orientation* if there exists a surjection $\theta : L \oplus A^{d-1} \twoheadrightarrow J$, such that ω is the induced isomorphism

$$L/JL \overset{\alpha}{\xrightarrow{\sim}} \wedge^n(L/JL \oplus (A/J)^{d-1}) \overset{\wedge^n(\theta)}{\xrightarrow{\sim}} \wedge^d(J/J^2)$$

where $\alpha(\bar{e}) = \bar{e} \wedge \bar{e_2} \wedge \ldots \wedge \bar{e_d}$ (and $\{e_2, e_3, \ldots, e_d\}$ is a basis of A^{d-1}).

Let H be the subgroup of G generated by the set of pairs (J, ω_J), where J is a finite intersection of maximal ideals of height d and ω_J is a global L-orientation of J.

Definition 1.1. The Euler class group of A with respect to L is $\mathrm{E}(A, L) = G/H$. In case $L = A$, we write $E(A)$ for $E(A, A)$.

Further, let G_0 be the free abelian group on the set $(\mathcal{M})$ where $\mathcal{M}$ is a maximal ideal of A. Let $J = \cap_{i=1}^n \mathcal{M}_i$ be a finite intersection of maximal ideals. Let (J) denote the element $\sum_i (\mathcal{M}_i)$ of G_0. Let H_0 be the subgroup of G_0 generated by elements of the type (J), where J is a finite intersection of maximal ideals such that there exists a surjection $\alpha : L \oplus A^{d-1} \twoheadrightarrow J$.

Definition 1.2. The weak Euler class group of A is $\mathrm{E}_0(A, L) := G_0/H_0$.

From the definitions of $\mathrm{E}(A,L)$ and $\mathrm{E}_0(A,L)$, it is clear that there is a canonical surjection $\mathrm{E}(A,L) \twoheadrightarrow \mathrm{E}_0(A,L)$.

Now let P be a projective A-module of rank d such that $L \simeq \wedge^d(P)$ and let $\chi : L \xrightarrow{\sim} \wedge^d P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is a finite intersection of maximal ideals of height d. Therefore we obtain an induced isomorphism $\overline{\varphi} : P/JP \to J/J^2$. Let ω_J be the local L-orientation of J given by $\wedge^d(\overline{\varphi}) \circ \overline{\chi}$. Let $e(P,\chi)$ be the image in $E(A,L)$ of the element (J,ω_J) of G. The assignment sending the pair (P,χ) to the element $e(P,\chi)$ of $E(A,L)$ can be shown to be well defined from [5, Section 4].

Definition 1.3. The *Euler class* of (P,χ) is defined to be $e(P,\chi)$.

As mentioned earlier, the Euler class is a precise obstruction, i.e. with the above set-up, $P \simeq Q \oplus A$ for some projective A-module Q of rank $d-1$ if and only if $e(P,\chi) = 0$ in $\mathrm{E}(A,L)$ [5, Corollary 4.4].

Much before these definitions were made, Murthy [17, Theorem 3.8] proved that when the base field k is algebraically closed, $c_d(P) = 0$ is also sufficient for the projective module to split off a free summand of rank 1. There is a natural map $\psi : \mathrm{E}(A,L) \twoheadrightarrow CH^d(A)$ and under this map $e(P,\chi)$ gets mapped to $c_d(P)$. Thus, when $c_d(P) = 0$, $e(P,\chi)$ is in the kernel of the map ψ.

When the base field k is algebraically closed, $\mathrm{E}(A,L) \xrightarrow{\sim} CH^d(X)$ for any invertible module L. Subsequently, using the notion of the Euler class group, it was shown in [4] and [3] that when d is odd and the base field k is $\mathbb{R}$, $e(P,\chi) = 0$ if and only if $c_d(P) = 0$.

2. Chow-Witt groups

In this section, we refer to [2] and [8] for more information on Chow-Witt groups. Let F be a field and L be a F-vector space of dimension 1. Let $W(F,L)$ be the Witt group of non-degenerate symmetric bilinear forms with coefficients in L. The tensor product induces a structure of $W(F)$-module on $W(F,L)$ where $W(F) := W(F,F)$ is the classical Witt ring. Let $I(F) \subset W(F)$ be the fundamental ideal of even dimensional symmetric bilinear forms, and $I^n(F)$ its n^{th}-power for $n \in \mathbb{N}$. For convenience of notations, we also set $I^n(F) = W(F)$ for $n \leq 0$. We denote by $I^n(F,L)$ the group $I^n(F) \cdot W(F,L)$. By definition, $I^{n+1}(F,L) \subset I^n(F,L)$ for any $n \in \mathbb{Z}$ and we set $\overline{I}^n(F) := I^n(F,L)/I^{n+1}(F,L)$. As suggested by the notation, the group $\overline{I}^n(F)$ is completely independent of L.

For any $n \in \mathbb{N}$, let $K_n^M(F)$ the the n-th Milnor K-theory group of F as defined in [15]. We set $K_n^M(F) = 0$ if $n < 0$. Recall from [15, Theorem 4.1] that for any $n \in \mathbb{Z}$ there is a homomorphism $s_n : K_n^M(F) \to \overline{I}^n(F)$ defined on symbols by $s_n(\{a_1, \dots, a_n\}) = \langle 1, -a_1 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$. We define the group $G^n(F,L)$ as the fibre product

$$\begin{array}{ccc} G^n(F,L) & \longrightarrow & I^n(F,L) \\ \downarrow & & \downarrow \pi_n \\ K_n^M(F) & \xrightarrow[s_n]{} & \overline{I}^n(F) \end{array}$$

where $\pi_n : I^n(F,L) \to \overline{I}^n(F)$ is the projection.

Let A be a discrete valuation ring and L be an invertible A-module. Let F be the quotient field of A and k be its residue field. For any $n \in \mathbb{Z}$, there is a residue homomorphism $d_K^n : K_n^M(F) \to K_{n-1}^M(k)$ ([15, Lemma 2.1] and a residue homomorphism $d_I^n : I^n(F,L) \to I^{n-1}(k, \mathrm{Ext}^1_A(k,L))$ ([8, Chapter 7, Chapter 9]) which induce the same homomorphism $\overline{I}^n(F) \to \overline{I}^{n-1}(k)$ ([8, Proposition 10.2.5]). Therefore we get a residue homomorphism $d_G^n : G^n(F,L) \to G^{n-1}(k, \mathrm{Ext}^1_A(k,L))$.

Let X be an integral regular scheme over a field k and let $U \subset X$ be an open subset. Let L be a line bundle over X. Any point $x \in U^{(1)}$ yields a discrete valuation on $k(U) = k(X)$ and we get for any $n \in \mathbb{Z}$ a homomorphism

$$d_G^n : G^n(k(X), L) \to \bigoplus_{x \in U^{(1)}} G^{n-1}(k(x), \omega_x \otimes L).$$

Associating $G^n(U,L) := \ker d_G^n$ to $U \subset X$ defines a presheaf on X and we consider its associated sheaf $\mathcal{G}_L^n$ (in the Zariski topology).

Definition 2.1. Let X be a regular scheme over a field k and let L be a line bundle over X. For any $n \in \mathbb{Z}$ the Chow-Witt group $\widetilde{CH}^n(X,L)$ is defined as the cohomology group $H^n(X, \mathcal{G}_L^n)$. We denote by $\widetilde{CH}^n(X)$ the group $\widetilde{CH}^n(X, \mathcal{O}_X)$.

Let $\mathcal{K}_n^M$ be the Zariski sheaf associated to the presheaf $U \mapsto K_n^M(U) := \ker d_K^n$, where d_K^n is defined for Milnor K-theory in the same way as d_G^n is defined above. Then the homomorphism $G^n(k(X), L) \to K_n^M(k(X))$ induces a morphism of sheaves $\mathcal{G}_L^n \to \mathcal{K}_n^M$. By Bloch's formula, we have $H^n(X, \mathcal{K}_n^M) = CH^n(X)$ and therefore there is a natural homomorphism $\widetilde{CH}^n(X,L) \to CH^n(X)$ for any $n \in \mathbb{Z}$.

In a similiar fashion, we can associate a sheaf to the unramified elements in $I^n(k(X), L)$, which we denote by $\mathcal{I}_L^n$. By definition of the sheaves, we get an exact sequence

$$0 \longrightarrow \mathcal{I}_L^{n+1} \longrightarrow \mathcal{G}_L^n \longrightarrow \mathcal{K}_n^M \longrightarrow 0$$

which is useful for computations.

In case $X = \mathrm{Spec}(A)$ is an affine (regular) scheme of dimension d over a field k, the Chow-Witt group of closed points $\widetilde{CH}^d(X)$ admits a presentation by generators and relations ([8, Theorem 10.3.5]). It follows immediately that there is a surjective homomorphism $\eta : E(A) \to \widetilde{CH}^d(X)$ ([8, Proposition 17.2.8]). The question to know whether this homomorphism is an isomorphism is still open. It is known to be the case when X is a surface over a field of characteristic 0 ([8, Corollary 17.4.2]), or when X is of dimension d over $\mathbb{R}$ and oriented ([8, Corollary 17.4.6]).

3. The example

Let $k = \mathbb{Q}(i)(t_1, t_2, \ldots, t_n)$, where $n \geq 3$. Consider the Pfister quadratic form $q := \langle 1, -t_1, -t_2, t_1t_2 \rangle$ on k and the associated smooth quadric Q in $\mathbb{P}^3$. It is easy to check that q is anisotropic on k (use for instance the residues associated to the t_1 and t_2 valuations defined in [15, Corollary 5.1]).

Let $U = \mathbb{P}^3 - Q$ be the open complement of Q. Then U is affine, and we set $U = \mathrm{Spec}(A)$ with $A = \mathcal{O}_U(U)$. Our goal in this section is to prove that the kernel of the homomorphism $\widetilde{CH}^3(U) \to CH^3(U)$ is a non trivial 2-torsion abelian group. We begin with the well-known computation of $CH^3(U)$.

Lemma 3.1. *We have $CH^3(U) = \mathbb{Z}/2$.*

Proof. Recall first that the push-forward homomorphism $p_*: CH^3(\mathbb{P}^3) \to \mathbb{Z}$ is an isomorphism ([10, Chapter 3]). The exact sequence

$$CH^2(Q) \longrightarrow CH^3(\mathbb{P}^3) \longrightarrow CH^3(U) \longrightarrow 0$$

of [10, Proposition 1.8] becomes then an exact sequence

$$CH^2(Q) \xrightarrow{p'_*} \mathbb{Z} \longrightarrow CH^3(U) \longrightarrow 0$$

where $p': Q \to \mathrm{Spec}(k)$ is the projection. A closed point on Q corresponds to a field F such that the quadratic form $q := \langle 1, -t_1, -t_2, t_1t_2 \rangle$ is isotropic on F. By Springer's theorem [14, VII, Theorem 2.3], the extension of fields $[F:k]$ is of even degree. On the other hand, it is clear that there exists a field F of degree 2 such that q is isotropic on F. Thus $CH^3(U) = \mathbb{Z}/2$. □

We want now to estimate $\widetilde{CH}^3(U)$. As for Chow groups, we use the exact sequence associated to an open embedding ([8, Corollary 10.4.9])

$$H^3_Q(\mathbb{P}^3, \mathcal{G}^3) \longrightarrow \widetilde{CH}^3(\mathbb{P}^3) \longrightarrow \widetilde{CH}^3(U) \longrightarrow 0.$$

To compute $\widetilde{CH}^3(\mathbb{P}^3)$, recall first that since $\mathcal{O}(-4) = \mathcal{O}(-2) \otimes \mathcal{O}(-2)$ there is a natural isomorphism $\widetilde{CH}^3(\mathbb{P}^3) \to \widetilde{CH}^3(\mathbb{P}^3, \mathcal{O}(-4))$. Now $\mathcal{O}(-4)$ is the canonical bundle of $\mathbb{P}^3$ and we therefore obtain a push-forward homomorphism ([8, Corollary 10.4.5])

$$p_*: \widetilde{CH}^3(\mathbb{P}^3, \mathcal{O}(-4)) \to \widetilde{CH}^0(k) = GW(k).$$

where $p: \mathbb{P}^3 \to \mathrm{Spec}(k)$ is the projection. We still denote by p_* the composition $\widetilde{CH}^3(\mathbb{P}^3) \to \widetilde{CH}^3(\mathbb{P}^3, \mathcal{O}(-4)) \to GW(k)$.

Proposition 3.2. *The push-forward homomorphism $\widetilde{CH}^3(\mathbb{P}^3) \to GW(k)$ is an isomorphism.*

Proof. First observe that the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}^4 \longrightarrow \mathcal{G}^3 \longrightarrow \mathcal{K}_3^M \longrightarrow 0$$

induces a long exact sequence of localization

$$\ldots \longrightarrow H^2(\mathbb{P}^3, \mathcal{K}_3^M) \xrightarrow{\delta} H^3(\mathbb{P}^3, \mathcal{I}^4) \xrightarrow{h} H^3(\mathbb{P}^3, \mathcal{G}^3) \longrightarrow H^3(X, \mathcal{K}_3^M) \longrightarrow 0.$$

The push-forward homomorphisms give by definition a commutative diagram with exact lines

$$\begin{array}{ccccccccc}
 & & H^3(\mathbb{P}^3, \mathcal{I}^4) & \xrightarrow{h} & H^3(\mathbb{P}^3, \mathcal{G}^3) & \longrightarrow & H^3(X, \mathcal{K}_3^M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\
0 & \longrightarrow & H^0(k, \mathcal{I}) & \longrightarrow & H^0(k, \mathcal{G}^0) & \longrightarrow & H^0(k, \mathcal{K}_0^M) & \longrightarrow & 0.
\end{array}$$

It follows from [9, Theorem 9.4] that the left p_* is an isomorphism and therefore h is also injective. The right hand p_* is also an isomorphism by [10, Theorem 3.3(b)] and the five lemma allows to conclude. □

The image of the push-forward isomorphism $\widetilde{CH}^3(\mathbb{P}^3) \to GW(k)$ of Proposition 3.2 is given by the images of the trace maps $Tr_x : GW(k(x)) \to GW(k)$ where $x \in \mathbb{P}^3$ is a closed point. We briefly recall the definition of the trace maps, or more generally of the transfer maps, since we use them in the sequel.

For a finite extension of fields $K \subset L$ (say, of dimension n), choose any non-zero K-linear map $s : L \to K$. Then one defines a transfer map $s_* : GW(L) \to GW(K)$ as follows :
Let $e_i, 1 \leq i \leq n$ be a basis of L over K and $a \in L^\times$. Consider the map sending $q : (e_i, e_j) \mapsto s(ae_ie_j)$. This defines a symmetric bilinear form on a n-dimensional vector space over K. It can be checked that this is non-degenerate ([14, VII, Proposition 1.1]). Then define the map $s_* : GW(L) \to GW(K)$ by sending the form $\langle a\rangle \in GW(L)$ to the form q defined above. It turns out that the image is independent of the choice of s ([14, Remark (C), p. 194]).

When the extension is separable, one can choose s to be the trace map, in which case the corresponding map of Grothendieck-Witt groups is called the trace form and denoted by $Tr_{L/K}$.

Observe that for any non trivial $s : L \to K$ the corresponding homomorphism s_* on Grothendieck-Witt groups is $GW(K)$-linear.

In view of Proposition 3.2, we get a presentation

$$\bigoplus_{x\in Q^{(2)}} GW(k(x)) \xrightarrow{\sum(s_x)_*} GW(k) \longrightarrow \widetilde{CH}^3(U) \longrightarrow 0$$

with $s_x : k(x) \to k$ any nonzero k-homomorphism.

Lemma 3.3. *The group* $\widetilde{CH}^3(U)$ *is* 2*-torsion.*

Proof. Consider the fields $L_1 = k(\sqrt{t_1}), L_2 = k(\sqrt{t_2}), L_3 = k(\sqrt{t_1t_2})$ which are all degree 2 extensions of k and correspond to closed points on Q (for L_3 recall that -1 is a square in k). For any $1 \leq i \leq 3$, define a k-homomorphism $s_i : L_i \to k$ by $s_i(1) = 1$ and $s_i(\sqrt{t_i}) = 0$. We observe then that $(s_i)_*(\langle 1\rangle) = \langle 1, t_i\rangle$ and therefore

$$(s_1)_*(\langle 1\rangle) + (s_3)_*(\langle 1\rangle) - (s_2)_*(\langle t_1\rangle) = \langle 1, t_1\rangle + \langle 1, t_1t_2\rangle - \langle t_1, t_1t_2\rangle = \langle 1, 1\rangle.$$

Thus $\widetilde{CH}^3(U)$ is 2-torsion. $\square$

Consider the group

$$\begin{aligned} G_k(q) &:= \{a \in k^\times | aq \simeq q\} \\ &= \{a \in k^\times | aq = q \in W(k)\} \\ &= \{a \in k^\times | < 1, -a > \otimes q = 0 \in W(k)\}. \end{aligned}$$

Clearly $(k^\times)^2 \subseteq G_k(q) \subseteq k^\times$ and hence we have a short exact sequence

$$0 \longrightarrow \frac{G_k(q)}{(k^\times)^2} \longrightarrow \frac{k^\times}{(k^\times)^2} \longrightarrow \frac{k^\times}{G_k(q)} \longrightarrow 0.$$

Now the determinant map $\det : GW(k) \to \frac{k^\times}{(k^\times)^2}$ gives us a diagram with exact bottom row

$$\begin{array}{ccccccccc} & & & & GW(k) & & & & \\ & & & & \downarrow{\scriptstyle \det} & & & & \\ 0 & \longrightarrow & \frac{G_k(q)}{(k^\times)^2} & \longrightarrow & \frac{k^\times}{(k^\times)^2} & \longrightarrow & \frac{k^\times}{G_k(q)} & \longrightarrow & 0 \end{array}$$

Remark 3.4. Note that since -1 is a square in k, the determinant map coincides with the more classical discriminant map.

Lemma 3.5. *Let L/k be a finite separable extension and let $s : L \to k$ be any non-zero linear map. Then the image of the composition*

$$GW(L) \xrightarrow{s_*} GW(k) \xrightarrow{\det} \frac{k^\times}{(k^\times)^2}$$

is contained in the image of $\bar{N}_{L/k} : L^\times \to \frac{k^\times}{(k^\times)^2}$ induced by the norm map.

Proof. Let $r = [L : k]$. The image of s_* being independent of s, we can choose any non-zero s to prove the lemma. Since -1 is a square in $k^\times$, the determinant induces a map

$$\det : W(k) \to \frac{k^\times}{(k^\times)^2}$$

and it suffices to prove that the image of the composition

$$W(L) \xrightarrow{s_*} W(k) \xrightarrow{\det} \frac{k^\times}{(k^\times)^2}$$

is contained in the image of $\bar{N}_{L/k}$ to conclude.

Since L is finite and separable, the primitive element theorem gives us that $L = k(y)$. Then $1, y, y^2, \ldots, y^{r-1}$ is a k-basis of L as a vector space. Hence, choosing $s(1) = 1$ and $s(y^i) = 0, i = 1, 2, \ldots, r-1$, we get a non-zero k-linear transformation. For this transformation, Scharlau [14, VII, Theorem 1.6] has computed the transfer map s_*, namely

$$s_*(\langle 1\rangle) = \begin{cases} \langle 1\rangle & r = 2m+1 \\ \langle 1, -N_{L/k}(y)\rangle & r = 2m. \end{cases}$$

Since $-1 \in (k^\times)^2$, it follows that $\det(s_*(\langle 1\rangle)) \in \bar{N}_{L/k}(L^\times)$. Since $\langle -1\rangle = \langle 1\rangle$ in $W(L)$, the same is true of $\langle -1\rangle$.

Now, for any $a \in L^\times$, we have $\langle -a\rangle = \langle 1, -a\rangle + \langle -1\rangle$ and therefore

$$\det(s_*(\langle -a\rangle)) = \det(s_*(\langle 1, -a\rangle))\det(s_*(\langle -1\rangle)).$$

So it is enough to prove that $\det(s_*(\langle 1, -a\rangle)) \in \bar{N}_{L/k}(L^\times)$. However, these are exactly the generators of the fundamental ideal $I(L) \subset W(L)$ and it is known from [8, Lemma 10.2.3] that there is a commutative diagram

$$\begin{array}{ccc} I(L) & \xrightarrow{Tr_{L/k}} & I(k) \\ {\scriptstyle \det}\downarrow & & \downarrow{\scriptstyle \det} \\ \frac{L^\times}{(L^\times)^2} & \xrightarrow[\bar{N}_{L/k}]{} & \frac{k^\times}{(k^\times)^2}. \end{array}$$

Now [14, Remark (C), p. 194] shows that $s_*(\_) = Tr_{L/k}(\langle b\rangle \cdot \_)$ for some $b \in L^\times$. Since any $\alpha \in I(L)$ is of even rank, we have $\det(\alpha) = \det(\langle b\rangle \cdot \alpha)$ and then

$$\det(s_*(\alpha)) = \det(Tr_{L/k}(\langle b\rangle \cdot \alpha)) = \bar{N}_{L/k}(\det(\langle b\rangle \cdot \alpha)) = \bar{N}_{L/k}(\det(\alpha)).$$

□

We now deal with the composition $GW(k) \xrightarrow{\det} \frac{k^\times}{(k^\times)^2} \to \frac{k^\times}{G_k(q)}$.

Lemma 3.6. *Let $x \in Q$ be a closed point and $s_x : k(x) \to k$ be a nonzero k-homomorphism. Then the composition*

$$GW(k(x)) \xrightarrow{(s_x)_*} GW(k) \xrightarrow{\det} \frac{k^\times}{(k^\times)^2} \longrightarrow \frac{k^\times}{G_k(q)}$$

is trivial.

Proof. Note that Lemma 3.5 implies that $\det((s_x)_*(GW(k(x)))) \subseteq \bar{N}_{k(x)/k}(k(x)^\times)$. Hence, if we prove that $N_x(k(x)^\times) \subset G_k(q)$, we will have proved the lemma. But observe that for every point $x \in Q$, the form $q_{k(x)}$ is isotropic, hence hyperbolic (since it is a Pfister form). An easy consequence of Scharlau's Norm Principle [14, VII, Corollary 4.5] gives us $N_{k(x)/k}(k(x)^\times) \in G_k(q)$ for all $x \in V$. Hence, the lemma stands proved. □

Proposition 3.7. *For any $3 \leq j \leq n$, the element $\langle -1, t_j\rangle \in \widetilde{CH}^3(\mathbb{P}^3)$ gives a non trivial element in $\widetilde{CH}^3(U)$.*

Proof. The presentation

$$\bigoplus_{x \in Q^{(2)}} GW(k(x)) \xrightarrow{\sum (s_x)_*} GW(k) \longrightarrow \widetilde{CH}^3(U) \longrightarrow 0$$

shows that we have to prove that $\langle -1, t_j\rangle$ is not in the image of $\sum(s_x)_*$. Now Lemma 3.6 shows that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \bigoplus_{x \in V} GW(k(x)) & \xrightarrow{\sum (s_x)_*} & GW(k) & \longrightarrow & \widetilde{CH}^3(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \frac{G_k(q)}{(k^*)^2} & \longrightarrow & \frac{k^*}{(k^*)^2} & \longrightarrow & \frac{k^*}{G_k(q)} & \longrightarrow & 0
\end{array}$$

Thus, it is enough to prove that $\det(\langle -1, t_j\rangle)$ is non-zero in $\frac{k^*}{G_k(q)}$, which amounts to show that $t_j \notin G_k(q)$. But $\langle 1, -t_1\rangle \otimes \langle 1, -t_2\rangle \otimes \langle 1, -t_j\rangle \neq 0$ in $W(k)$ as the successive use of the residue maps in the t_1, t_2 and t_j-valuations show. □

As a corollary, we get the following theorem.

Theorem 3.8. *Let $k = \mathbb{Q}(i)(t_1, t_2, \ldots, t_n)$ with $n \geq 3$. Consider the Pfister form $q := \langle 1, -t_1, -t_2, t_1t_2\rangle$ and its associated smooth quadric Q in $\mathbb{P}^3$. Let $U = \mathbb{P}^3 - Q$ be the open (affine) complement of Q and let $u \in U$ be a closed rational point. The element $\langle -1, t_j\rangle \in \widetilde{CH}^3_u(U)$ yields a non trivial 2-torsion element in the kernel of the homomorphism $\widetilde{CH}^3(U) \to CH^3(U)$.*

Proof. In view of Proposition 3.7, we know that $\langle -1, t_j\rangle$ is non trivial. Moreover, $\widetilde{CH}^3(U)$ is 2-torsion by Lemma 3.3 and it suffices to check that $\langle -1, t_j\rangle$ vanishes under the homomorphism

$$\widetilde{CH}^3(U) \to CH^3(U)$$

to conclude. But $\langle -1, t_j\rangle$ is supported on a vector-space of dimension 2 and Lemma 3.1 gives the result. □

Remark 3.9. One can in fact prove that the elements $\langle -1, t_j\rangle$ for $3 \le j \le n$ are independent in $\widetilde{CH}^3(\mathbb{P}^3)$, and thus the kernel of $\widetilde{CH}^3(U) \to CH^3(U)$ is an $\mathbb{F}_2$-vector space of dimension $\geq n-2$.

4. 2-torsion in the Euler class group

Let $A = \mathcal{O}_U(U)$ where U is as in the previous section. In this section, we use Theorem 3.8 to produce an element which is 2-torsion in the Euler class group $E(A)$ and vanishes in $E_0(A)$.

To start with, recall from [8, Proposition 17.2.8] that there is a natural surjective homomorphism

$$\eta : E(A) \to \widetilde{CH}^3(A).$$

Consider the closed point $x := [1 : 0 : 0 : 0]$ of $\mathbb{P}^3$ and the corresponding maximal ideal $\mathfrak{m}$ of A. The orientation class of [9, §6] (see also [1, Definition 7.3]) yields an isomorphism $k \simeq \mathrm{Ext}^3_{\mathcal{O}_{\mathbb{P}^3}}(i_*\mathcal{O}_x, \mathcal{O}_{\mathbb{P}^3})$ and thus an orientation $\omega_\mathfrak{m}$ of $\mathfrak{m}$. Let $\alpha := (\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -t_3\omega_\mathfrak{m}) \in E(A)$.

Then under the above map, $\alpha \mapsto \langle 1, -t_3\rangle \in \widetilde{CH}^3(A)$. Since the image of α is non-zero in $\widetilde{CH}^3(A)$, α is non-zero in $E(A)$. We will now prove it is 2-torsion. To show that, note that

$$\begin{aligned}
2\alpha &= (\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -t_3\omega_\mathfrak{m}) + (\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -t_3\omega_\mathfrak{m}) \\
&= (\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -t_3\omega_\mathfrak{m}) + (\mathfrak{m}, -\omega_\mathfrak{m}) + (\mathfrak{m}, t_3\omega_\mathfrak{m}) \\
&= [(\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -\omega_\mathfrak{m})] + [(\mathfrak{m}, -t_3\omega_\mathfrak{m}) + (\mathfrak{m}, t_3\omega_\mathfrak{m})]
\end{aligned}$$

By [3, Lemma 3.6], we have

$$(\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -\omega_\mathfrak{m}) = (\mathfrak{m}, -t_3\omega_\mathfrak{m}) + (\mathfrak{m}, t_3\omega_\mathfrak{m})$$

and further, $(\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -\omega_\mathfrak{m})$ is the image of $(\mathfrak{m}) \in E_0(A)$ under the homomorphism $\theta : E_0(A) \to E(A)$ as in [3, Proposition 3.7]. Then, putting it all together, we get that

$$\theta(2(\mathfrak{m})) = 2\theta((\mathfrak{m})) = 2\alpha.$$

Hence, it is enough to prove that $2(\mathfrak{m}) = 0$ in $E_0(A)$.

Let $Y = V(X_3) \cap X$. Since X is affine, $Y = V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of height 1 of A. Since A is a regular domain, $\mathfrak{p}$ is an invertible ideal, hence isomorphic to a projective A-module L of rank 1. Let $B = A/\mathfrak{p}$. Then there exists a canonical group homomorphism $\gamma : E_0(B) \to E_0(A, L)$ given by $(I) \mapsto (J)$ where J is the ideal of A containing $\mathfrak{p}$ such that $I = J/\mathfrak{p}$. Further, we know by [5, Theorem 6.8] that $\psi : E_0(A, L) \xrightarrow{\sim} E_0(A)$. Note that $\mathfrak{m}$ contains the ideal $\mathfrak{p}$. Hence, letting $\mathfrak{n}$ be the ideal $\mathfrak{m}/\mathfrak{p}$ of B, we have

$$\theta(\psi(\gamma(2(\mathfrak{n})))) = 2\theta((\mathfrak{m})) = 2\alpha.$$

Hence, it is enough to prove that $2(\mathfrak{n})$ is 0 in $E_0(B)$.

Since $\mathrm{Spec}(B) = \mathbb{P}^2_k \setminus V(G)$ where $G(X_0, X_1, X_2) = {X_0}^2 - t_1{X_1}^2 - t_2{X_2}^2$, we see that B is an open set obtained as complement of a degree 2 hypersurface with no k-rational points. Now the exact sequence

$$CH^2(V(G)) \to CH^2(\mathbb{P}^2_k) \to CH^2(B) \to 0$$

gives us $CH^2(B) = \mathbb{Z}/2$ as in Lemma 3.1. Further, since B is 2-dimensional, we know that $E_0(B) \simeq CH^2(B)$ (this follows in particular from the results in [6]). Hence, $2(\mathfrak{n}) = 0$. Hence, we get that the non-zero element α of $E(A)$ is indeed 2-torsion in $E(A)$. We have thus proved:

Theorem 4.1. *Let $k = \mathbb{Q}(i)(t_1, t_2, \ldots, t_n)$ with $n \geq 3$. Consider the Pfister form $q := \langle 1, -t_1, -t_2, t_1t_2\rangle$ and its associated smooth quadric Q in $\mathbb{P}^3$. Let $U = \mathbb{P}^3 - Q$ be the open (affine) complement of Q and $A = \mathcal{O}_U(U)$. Let $x := [1 : 0 : 0 : 0] \in U$ and $\mathfrak{m} \subset A$ be the corresponding maximal ideal. Then there exists a local orientation $\omega_{\mathfrak{m}}$ of $\mathfrak{m}$ such that $(\mathfrak{m}, \omega_{\mathfrak{m}}) + (\mathfrak{m}, -t_3\omega_{\mathfrak{m}})$ is 2-torsion in $E(A)$ and vanishes under the homomorphism $E(A) \to E_0(A)$.*

5. Euler classes

Let A be the ring considered in the previous section. We prove that any projective A-module of rank 3 with trivial determinant admits a free factor of rank one.

The universal submodule $\mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^3}$ yields an exact sequence of locally free $\mathcal{O}_{\mathbb{P}^3}$-modules

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{i} \mathcal{O}^4_{\mathbb{P}^3} \longrightarrow G \longrightarrow 0$$

where G is the quotient. The dual of this sequence reads as

$$0 \longrightarrow G^\vee \longrightarrow (\mathcal{O}^4_{\mathbb{P}^3})^\vee \xrightarrow{i^\vee} \mathcal{O}(1) \longrightarrow 0.$$

We can endow $\mathcal{O}^4_{\mathbb{P}^3}$ with the usual skew-symmetric isomorphism $\psi : \mathcal{O}^4_{\mathbb{P}^3} \to (\mathcal{O}^4_{\mathbb{P}^3})^\vee$. Since the only skew-symmetric morphism between a line bundle and its dual is trivial, it follows that $i^\vee\psi i = 0$ and therefore we get the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-1) & \xrightarrow{i} & \mathcal{O}^4_{\mathbb{P}^3} & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \psi & & \downarrow f & & \\
0 & \longrightarrow & G^\vee & \longrightarrow & (\mathcal{O}^4_{\mathbb{P}^3})^\vee & \xrightarrow[i^\vee]{} & \mathcal{O}(1) & \longrightarrow & 0.
\end{array}$$

Let $F = \ker f$. The Snake lemma shows that F is endowed with a skew-symmetric isomorphism $\varphi : F \to F^\vee$ and it follows that F is of trivial determinant. We denote by F_A (resp. G_A) the restriction of F (resp. G) to A.

Lemma 5.1. *The Chow group $CH^2(A)$ is generated by $c_2(F_A)$.*

Proof. By definition of A, we have a surjective homomorphism $CH^2(\mathbb{P}^3) \to CH^2(A)$. It suffices thus to prove that $c_2(F)$ generates $CH^2(\mathbb{P}^3)$ to conclude. Now $CH^2(\mathbb{P}^3)$ is generated by $c_1(\mathcal{O}(-1))^2$ and a simple computation using Whitney formula ([10, Chapter 3, Theorem 3.2(e)]) on the exact sequences

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{i} \mathcal{O}^4_{\mathbb{P}^3} \longrightarrow G \longrightarrow 0$$

and

$$0 \longrightarrow F \longrightarrow G \xrightarrow{f} \mathcal{O}(1) \longrightarrow 0$$

yields $c_2(F) = c_1(\mathcal{O}(-1))^2$ (recall that $c_1(F) = 0$). □

Let $F^3K_0(A)$ be the subgroup of $K_0(A)$ generated by closed points. For any projective A-module P, we denote by $[P]$ its class in $K_0(A)$.

Lemma 5.2. *The group $F^3K_0(A)$ is generated by*

$$\beta := [G_A^\vee] + [\det {G_A}^\vee] - [\wedge^2 G_A^\vee] - [A].$$

Proof. Mapping a closed point to a locally free resolution of its structure sheaf yields a surjective homomorphism $CH^3(A) \to F^3K_0(A)$. Lemma 3.1 proves therefore that $F^3K_0(A)$ is generated by the class of a closed point. Consider the exact sequence

$$0 \longrightarrow G^\vee \longrightarrow (\mathcal{O}^4_{\mathbb{P}^3})^\vee \xrightarrow{i^\vee} \mathcal{O}(1) \longrightarrow 0.$$

Projecting $(\mathcal{O}^4_{\mathbb{P}^3})^\vee$ to the first factor yields a morphism $s : G^\vee \to \mathcal{O}_{\mathbb{P}^3}$. If we set $x := [1:0:0:0]$ then we get an exact sequence

$$G^\vee \xrightarrow{s} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_x \longrightarrow 0$$

and the Koszul complex associated to s is then a locally free resolution of $\mathcal{O}_x$. Thus the lemma is proved. □

Proposition 5.3. *Any projective A-module P of rank 3 with trivial determinant splits off a free factor of rank one.*

Proof. In view of [5, Theorem 7.13] it suffices to prove that $[P] = [Q \oplus A]$ in $K_0(A)$ with Q a projective module of rank 2 with trivial determinant. Any such module is endowed with a skew symmetric isomorphism and we consider the Grothendieck group $K_0Sp(A)$ of symplectic modules and the forgetful homomorphism

$$f : K_0Sp(A) \to K_0(A).$$

Using [7, Proposition 11], we see that any $\alpha \in \mathrm{Im}(f)$ is of the form $\alpha = [Q] + 2m[A]$ for some $m \in \mathbb{Z}$ and some projective module Q of rank 2 with trivial determinant. Hence $[P] - [A] \in \mathrm{Im}(f)$ if and only if $[P] = [Q \oplus A]$ in $K_0(A)$. Lemma 5.1 shows that there exists Q' of rank 2 and trivial determinant such that $[P] - [Q' \oplus A]$ has trivial Chern classes c_0, c_1 and c_2. It follows from the Riemann-Roch theorem without denominators in [12] that $[P] - [Q' \oplus A] \in F^3K_0(A)$. In view of Lemma 5.2, we are reduced to show that $\beta := [G_A^\vee] + [\det {G_A}^\vee] - [\wedge^2 G_A^\vee] - [A]$ is in $\mathrm{Im}(f)$. But $[G_A^\vee] = [F_A^\vee] - [\mathcal{O}(-1)_A]$ in $K_0(A)$ and therefore [11, Chapter 5, §1] yields $[\wedge^2 G_A^\vee] = [\wedge^2 F_A^\vee] + [F_A^\vee \otimes \mathcal{O}(-1)_A]$. Since $\det G^\vee = \mathcal{O}(-1)$ and $F^\vee$ has trivial determinant, we find

$$\beta = [F_A^\vee] - [F_A^\vee \otimes \mathcal{O}(-1)_A].$$

Both terms on the right hand side are classes of rank 2 modules with trivial determinant (observe that $\mathcal{O}(-2)_A \simeq A$), hence endowed with skew symmetric forms. Thus $\beta \in \mathrm{Im}(f)$. □

Corollary 5.4. *Any projective A-module P of rank 3 with trivial determinant satisfies $e(P, \chi) = 0$ for any orientation χ of P.*

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