# On Dévissage for Witt groups

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## 1 Introduction

In this article, we develop a version of the dévissage theorem [BW, Theorem 6.1] for Witt groups of Cohen-Macaulay rings. To introduce this theorem, let $A$ be a commutative Noetherian domain of dimension $d$ with 2 invertible and $K$ be its quotient field. It is a classical question (known as purity) as to when the complex

$$W(A) \to W(K) \to \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(A): ht(\mathfrak{p})=1} W(k(\mathfrak{p}))$$

is exact. Purity is a conjecture when $A$ is a regular local ring and is affirmatively settled in many cases.

In general, we can extend the above map to the right for any regular scheme by considering the Gersten-Witt complex. Let $X = \mathrm{Spec}(A)$ be of

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dimension $d$ with 2 invertible and $X^{(n)}$ denote the points of codimension $n$. A Gersten-Witt complex

$$0 \to W(A) \to \bigoplus_{x \in X^{(0)}} W(k(x)) \to \bigoplus_{x \in X^{(1)}} W(k(x)) \to \ldots \to \bigoplus_{x \in X^{(d)}} W(k(x)) \to 0$$

was first constructed by Pardon [Pa1]. He further conjectured the exactness of this sequence for regular local rings and affirmatively settled it in many cases.

Subsequently, with the introduction of triangular Witt groups by Balmer [DWG, TWGI, TWGII], Witt groups could be viewed as a cohomology theory. Using this, another Gersten-Witt complex could be defined (though both complexes look similar, it seems unproven that the differentials match) [BW], similar to the one in K-theory. Once again it is an open question as to when the complex is exact. In particular, it is conjectured to be so when $X = \mathrm{Spec}(A)$ where $A$ a regular, local ring and this is affirmatively settled in many cases. [DWG, TWGI, TWGII, BW] form the basic backbone of all the results in this article and we would like to remark that the interested reader would be well advised to take a look at them. *For any unexplained notations and definitions in the introduction, please refer to (2.1).*

The key result which allows one to move from derived Witt groups to the Gersten-Witt complex is dévissage [BW, Section 6] which states that for a regular, local ring $(A, \mathfrak{m}, k)$, of dimension $d$, we have $W^n(D^b_{fl}(A)) \cong W(k)$ if $n \equiv d \bmod 4$ and $W^n(D^b_{fl}(A)) \cong 0$ otherwise. We describe below the generalized form of this theorem for a Cohen-Macaulay ring $A$ with $\dim A_{\mathfrak{m}} = d$ for all maximal ideals $\mathfrak{m}$ and 2 invertible.

Suppose $A$ is a Cohen-Macaulay ring with $\dim A = d$. Since there are modules of infinite projective dimension over such a ring and of finite length, the usual duality $Ext^d(\_\_ , A)$ does not work well. The options are either to change the duality (w.r.t. the canonical module) but then use the category of all modules (coherent Witt groups) or impose finite projective dimension homology conditions on the complexes. The first path is taken and deeply studied in Pardon [Pa2, Pa3] and more recently in Gille [G1, G2], where they also establish a Gersten-Witt complex.

In this article, we take the second approach. Let $\mathcal{M}FPD(A)$ be the category of finitely generated $A-$modules with finite projective dimension, $\mathcal{M}FL(A)$ be the category of finitely generated $A-$modules with finite length, and $\mathcal{A} = \mathcal{M}FPD_{fl}(A)$ be the full subcategory of finitely generated $A-$modules with finite projective dimension and finite length (the "intersection"). Note then that $\mathcal{A}$ is an exact category and has a natural duality given by $M \mapsto Ext^d_A(M, A)$ and so we can consider the Witt group $W(\mathcal{A})$. By Balmer [TWGI], we already know that $W(\mathcal{A}) \cong W^d(D^b(\mathcal{A}))$.

We consider the category $D^b_{\mathcal{A}}(\mathcal{A})$ with homologies in $\mathcal{A}$. Based on the fact that $\mathcal{A}$ actually has the 2-out-of-3 property for objects, we prove that the duality actually restricts to $D^b_{\mathcal{A}}(\mathcal{A})$ and this allows a suitable definition of Witt groups $W^i(D^b_{\mathcal{A}}(\mathcal{A}))$. Once defined, we prove that the above isomorphism actually factors through isomorphisms $W(\mathcal{A}) \xrightarrow{\sim} W^d(D^b_{\mathcal{A}}(\mathcal{A})) \xrightarrow{\sim} W^d(D^b(\mathcal{A}))$.

We now consider the category $D^b_{\mathcal{A}}(\mathcal{P}(A))$. Similar to $D^b_{\mathcal{A}}(\mathcal{A})$, we establish that this category is closed under duality and define the Witt groups $W^i(D^b_{\mathcal{A}}(\mathcal{A}))$. Having done so, we prove our version of dévissage (5.12, 6.7) :

$$W(\mathcal{A}) \xrightarrow{\sim} W^d(D^b_{\mathcal{A}}(\mathcal{P}(A)))$$
$$W^-(\mathcal{A}) \xrightarrow{\sim} W^{d+2}(D^b_{\mathcal{A}}(\mathcal{P}(A)))$$
$$W^{d+1}(D^b_{\mathcal{A}}(\mathcal{P}(A))) \cong W^{d-1}(D^b_{\mathcal{A}}(\mathcal{P}(A))) \cong 0.$$

Note that when $A$ is regular, this is exactly the same theorem as that in [BW]. Since we prove the theorem without regularity assumptions, we do not have access to the equivalences of the derived categories with duality as in [BW] or [G1] (in particular we cannot use the powerful lemma of Keller [K, §1.5, Lemma and Example(b)]). Our proof is thus necessarily more elementary and naïve than the one in [BW]. One of the key ingredients in the proof is the construction of a special sublagrangian (5.6) for symmetric forms in $(D^b_{\mathcal{A}}(\mathcal{P}(A)))$. All our results are heavily dependent on the beautiful papers of Balmer [DWG, TWGI, TWGII] and we again remind the reader to take a look at them.

This result gives rise to the possibility of a Gersten-Witt complex for these Witt groups. Our interest in these studies also stems from the introduction of the Chow-Witt groups $\widetilde{CH}^r(A)$ for $0 \leq r \leq d$, due to Barge and Morel [BM] and developed by Fasel [F1], which serve as obstruction groups for splitting of projective modules (this might also serve as an explanation of our obsession with maintaining the category of projective modules in our statement of dévissage and the reluctance to move to coherent Witt groups).

Since there is a parallel theory of Euler class groups, which is defined over a Cohen-Macaulay ring, the eventual hope would be to define Chow-Witt groups also over any Cohen-Macaulay ring.

A word about the layout of the article : in section 2, we establish the basic definitions and a key result on projective dimensions. In the section 3, we establish the important theorem that the categories $D^b_{\mathcal{A}}(\mathcal{P}(A))$ and $D^b_{\mathcal{A}}(\mathcal{A})$ are closed under duality, and more specifically how the homologies of the dual look like. Once this is established, in section 4, we define the Witt groups of the above categories and expectedly, they are 4-periodic, i.e. $W^n(D^b_{\mathcal{A}}(\mathcal{P}(A))) \xrightarrow{\sim} W^{n+4}(D^b_{\mathcal{A}}(\mathcal{P}(A)))$. Finally, in sections 5 and 6,we prove our main theorems about dévissage.

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# 2 Basic Notations and Preliminaries

**Notations 2.1.** Throughout this article, $A$ **will denote a Cohen-Macaulay ring with** $\dim A_m = d \geq 2$**, for all maximal ideals** $m$ **of** $A$**. Further,** $2$ **is always invertible in** $A$**.** We set up the notations :

1. $\mathcal{M}(A)$ : category of finitely generated $A-$modules.
2. $\mathcal{M}FPD(A)$ : full subcategory of finitely generated $A-$modules with finite projective dimension.
3. $\mathcal{M}FL(A)$ : category of finitely generated $A-$modules with finite length.
4. $\mathcal{A} = \mathcal{M}FPD_{fl}(A)$ : category of finitely generated $A-$modules with finite length and finite projective dimension.
5. $\mathcal{P}(A)$ : category of finitely generated projective $A-$modules.

6. For any exact category $\mathcal{C}$, $Ch^b(\mathcal{C})$ is the category of bounded chain complexes with objects in $\mathcal{C}$, and $D^b(\mathcal{C})$ is its derived category.

7. For any two exact categories $\mathcal{C}, \mathcal{D}$ in an ambient abelian category $\mathcal{C}'$, $Ch^b_{\mathcal{D}}(\mathcal{C})$ is the full subcategory of $Ch^b(\mathcal{C})$ consisting of complexes with homologies in $\mathcal{D}$. $D^b_{\mathcal{D}}(\mathcal{C})$ is its derived category, which is also the full subcategory of $D^b(\mathcal{C})$ consisting of objects from $Ch^b_{\mathcal{D}}(\mathcal{C})$.

8. $\mathcal{R}$ : full subcategory of $D^b_{\mathcal{A}}(\mathcal{P}(A))$ consisting of objects $P_\bullet$ such that $P_i = 0$ for $i > d, i < 0$ and $H_i(P_\bullet) = 0$ for all $i \neq 0$ and $H_0(P_\bullet) \in \mathcal{A}$.

9. For objects $M$ in $\mathcal{A}$, let $M^\vee = Ext^d_A(M, A)$ and $\tilde{\varpi} : M \xrightarrow{\sim} M^{\vee\vee}$ be the identification by double ext. (but we make this more precise in diagram ((5)) and the explanation of $\iota$.)

10. We will denote complexes $P_\bullet$ by :
$$\cdots 0 \longrightarrow P_m \xrightarrow{\partial_m} P_{m-1} \longrightarrow \cdots\cdots \longrightarrow P_n \longrightarrow 0 \cdots$$

11. A non-zero complex $P_\bullet$ is said to be supported on $[m, n]$ if $P_i = 0$ for all $i < n$ and $i > m$.

12. For a complex $P_\bullet$ of projective $A-$modules $P^*_\bullet$ will denote the usual dual induced by $Hom(*, A)$ and $\varpi : P_\bullet \xrightarrow{\sim} P^{**}_\bullet$ will denote the identification by evaluation. Note that the degree $r-$component of the dual $P^*_\bullet$ is $(P_{-r})^*$.

13. The **length** of a non-zero complex $P_\bullet$ is defined as $\ell(P_\bullet) = u - l$ where $P_u \neq 0, P_l \neq 0$ and $P_i = 0$ for all $i < l$ and $i > u$.

14. Let $B_r = B_r(P_\bullet) := \partial_{r+1}(P_{r+1}) \subseteq P_r$ denote the module of $r-$boundaries and $Z_r = Z_r(P_\bullet) := \ker(\partial_r) \subseteq P_r$ denote the module of $r-$cycles (or the $r^{th}$ syzygy).

15. The $r^{th}-$homology of $P_\bullet$ will be denoted by $H_r = H_r(P_\bullet) := \frac{Z_r}{B_r}$. So, the $r^{th}-$homology of the dual is $H_r(P^*_\bullet) = \frac{\ker\left(\partial^*_{-(r-1)}\right)}{image(\partial^*_{-r})}$.

16. A full exact subcategory $\mathcal{C}$ of an abelian category $\mathcal{D}$ is said to have the 2-out-of-3 property if for every short exact sequence in $\mathcal{D}$, whenever two of the objects are objects of $\mathcal{C}$, then so is the third.

*Remark* 2.2. The subcategory $\mathcal{M}FPD(A)$ and $\mathcal{A}$ are both exact subcategories and in fact have the 2-out-of-3 property. The category $\mathcal{R}$ is also an exact category. Although it is a subcategory of $D^b_{\mathcal{A}}(\mathcal{P}(A))$, it has no translation and is actually naturally equivalent to the category $\mathcal{A}$. The natural functor $\eta : \mathcal{R} \to \mathcal{A}$ is given by sending a complex $Q_\bullet$ to $H_0(Q_\bullet)$. The inverse functor $\iota$ is given by associating to objects $M \in \mathcal{A}$ a projective resolution of length $d$.

Note further that when $A$ is not regular, the categories $D^b_{\mathcal{A}}(\mathcal{P}(A))$ and $D^b_{\mathcal{M}FPD(A)}(\mathcal{P}(A))$ are not closed under the cone operation as the following example demonstrates.

*Example* 2.3. Let $(A, \mathfrak{m})$ be a non-regular Cohen-Macauly ring with $\dim A = d$, such that $\mathfrak{m} = (f_1, f_2, \ldots, f_d, z)$. We can assume, using prime avoidance, that $f_1, f_2, \ldots, f_d$ is a regular sequence. Let $U_\bullet = Kos_\bullet(f_1, f_2, \ldots, f_d)$ be the Koszul complex. Since the only nonzero homology of $U_\bullet$ is $H_0(U_\bullet) = \frac{A}{(f_1, f_2, \ldots, f_d)} \in \mathcal{A}$, $U_\bullet$ and all its translates are objects of both the categories above. Let $C(z)$ denote the cone of the the chain complex map $z : U_\bullet \longrightarrow U_\bullet$. From the long exact homology sequence corresponding to the short exact sequence of chain complexes

$$0 \longrightarrow U_\bullet \longrightarrow C(z) \longrightarrow U_\bullet[1] \longrightarrow 0$$

it follows that

$$H_0(Cone(z)) \cong \operatorname{coker}\left(\frac{A}{(f_1, f_2, \ldots, f_d)} \xrightarrow{\cdot z} \frac{A}{(f_1, f_2, \ldots, f_d)}\right) \cong \frac{A}{\mathfrak{m}} \notin \mathcal{A}.$$

So, $C(z)$ is not an object in $D^b_{\mathcal{A}}(\mathcal{P}(A)$.

Next, we ask if $Ch^b_{\mathcal{M}FPD}(\mathcal{P}(A))$ is closed under duality. We thank Sankar Dutta for providing the following example :

**Example 2.4** (Dutta)**.** Let $(A, \mathfrak{m}, k)$ be any non-regular Cohen-Macaulay local ring, with $\dim A = d$. Let

$$\cdots \longrightarrow P_d \xrightarrow{\partial_d} P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow k \longrightarrow 0$$

be a projective resolution of $k$. Let $^*$ denote $Hom(-, A)$ and $M = cokernel(\partial_d^*)$. Since $Ext^r(k, A) = 0$ for all $0 \leq r < d$, the sequence

$$0 \longrightarrow P_0^* \longrightarrow \cdots \longrightarrow P_{d-1}^* \longrightarrow P_d^* \longrightarrow M \longrightarrow 0$$

is a projective resolution of $M$. Dualizing this sequence, it follows that $Ext_A^d(M, A) \cong k$, which does not have finite projective dimension. In particular, $Ch^b_{\mathcal{M}FPD}(\mathcal{P}(A))$ is not closed under duality.

Note however that in the above example, $M$ does not have finite length. Indeed, we will prove in section 3 that the category $Ch^b_{\mathcal{A}}(\mathcal{P}(A))$ is closed under duality.

We mention a few standard results for the sake of completeness.

**Lemma 2.5.** *Let $(A, m)$ be a Cohen-Macaulay local ring with $\dim A = d$. Let $M \in \mathcal{M}FL(A)$. Then Further, $Ext^i(M, A) = 0$ for all $i < d$. and $Ext^d(M, A) \neq 0$ is also in $\mathcal{M}FL(A)$. Note further that if $M \in \mathcal{A}$, then so is $Ext^d(M, A)$.*

**Lemma 2.6.** *Let $A$ be a Cohen-Macaulay ring with $\dim A = d$. Let $M \in \mathcal{A}$. There is a natural isomorphism*

$$\varpi : M \xrightarrow{\sim} M^{\vee\vee}.$$

**Corollary 2.7.** *$(\mathcal{A}, ^\vee, \pm\tilde{\varpi})$ are exact categories with duality.*

**Proposition 2.8.** *Let $A$ be Cohen-Macaulay with $d = \dim A_{\mathfrak{m}} \geq 2$ for all maximal ideals $\mathfrak{m}$. Let $P_\bullet$ be a complex in $Ch^b(\mathcal{P}(A))$. Assume that all the homologies $H_r := H_r(P_\bullet) \in \mathcal{A}$. Then we claim :*

1. *The modules $B_r$ and $Z_r$ have finite projective dimension $\forall r$. In that case, proj* $\dim(Z_r) =$ *proj* $\dim(B_{r-1}) - 1$.

2. *For all $r \in \mathbb{Z}$, we have proj* $\dim B_r \leq d-1$ *and so proj* $\dim Z_r \leq d-2$.

3. *If $H_r \neq 0$ then proj* $\dim B_r = d-1$.

**Proof.** Note that $P_\bullet$ is a bounded complex and so let it be supported on $[m,n]$. Then $Z_n = P_n$ and so has finite projective dimension. Since there are short exact sequences :

$$0 \to B_r \to Z_r \to H_r \to 0 \qquad 0 \to Z_r \to P_r \to B_{r-1} \to 0,$$

it is clear that if $Z_r$ is of finite projective dimension, then so are $B_r$ and $Z_{r+1}$, and hence the proof follows by induction. The second part of (1) also follows from the above exact sequence.

Since $B_r$ is torsion free, it has depth at least 1 and hence, by the Auslander-Buchsbaum theorem, $\operatorname{proj}\dim(B_r) \leq d-1$. So, $\operatorname{proj}\dim(Z_r) \leq d-2$. So, (2) is established.

Now, assume $H_r \neq 0$. Choose a maximal ideal $\mathfrak{m}$ in the support of $H_r$. Then, consider the localized short exact sequence

$$0 \to (B_r)_\mathfrak{m} \to (Z_r)_\mathfrak{m} \to (H_r)_\mathfrak{m} \to 0.$$

Then, we get a long exact sequence of $Tor_{A_\mathfrak{m}}(\_, A/\mathfrak{m})$, which gives us that

$$Tor^d_{A_\mathfrak{m}}((H_r)_\mathfrak{m}, A/\mathfrak{m}) \cong Tor^{d-1}_{A_\mathfrak{m}}((B_r)_\mathfrak{m}, A/\mathfrak{m}) \quad \text{and} \quad Tor^d_{A_\mathfrak{m}}((B_r)_\mathfrak{m}, A/\mathfrak{m}) \cong 0,$$

since we have already proved that *proj* $\dim_A Z_r \leq d-2$. Thus we obtain *proj* $\dim_{A_\mathfrak{m}}(B_r)_\mathfrak{m} = d-1$. Since we know that $\operatorname{proj}\dim_A(B_r) \leq d-1$, this implies $\operatorname{proj}\dim_A(B_r) = d-1$. This establishes (3). □

The complexes in $Ch^b(\mathcal{P}(A))$ with finite length homologies have at least $d$ nonzero components at the left where the homology is 0. This proposition plays a key role in sections 5 and 6.

**Proposition 2.9.** *Let $A$ be Cohen-Macaulay with $d = \dim A_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$. Let $P_\bullet$ be a bounded complex of projective modules, such that $H_i = 0\ \forall i > n$ and $H_n \neq 0$ is of finite length. Then $P_i \neq 0, n \leq i \leq n+d$.*

**Proof.** Since $H_i = 0 \ \forall i > n$ and the complex is bounded, we get that $\frac{P_n}{B_n}$ is of finite projective dimension, since the components with indices $\geq n$ give a resolution. Now, let $\mathfrak{m}$ be a maximal ideal in the support of $H_n(P_\bullet)$, then $(H_n(P_\bullet))_\mathfrak{m} \subseteq \frac{(P_n)_\mathfrak{m}}{(B_n)_\mathfrak{m}}$ is of finite length, and hence $\frac{(P_n)_\mathfrak{m}}{(B_n)_\mathfrak{m}}$ has depth 0. By the Auslander-Buchsbaum theorem, $\operatorname{proj}\dim_{A_\mathfrak{m}}(\frac{(P_n)_\mathfrak{m}}{(B_n)_\mathfrak{m}}) = d$. But that means $\operatorname{proj}\dim_A(\frac{P_n}{B_n}) = d$. Hence, the resolution of $\frac{P_n}{B_n}$ given by the components of $P_\bullet$ with indices $\geq n$ must have length at least $d$. Hence, $P_i \neq 0, n \leq i \leq n+d$. The proof is complete. $\square$

# 3 Duality

As always, $A$ **will denote a Cohen-Macaulay ring with** $\dim A_m = d \geq 2$**, for all maximal ideals** $m$ and $\mathcal{A} = \mathcal{M}FPD_{fl}(A)$. In this section, we prove that the category $Ch^b_\mathcal{A}(\mathcal{P}(A))$ is closed under duality and give a precise description of the homologies of the dual.

**Theorem 3.1.** *Suppose $P_\bullet$ is a complex in $Ch^b(\mathcal{P}(A))$ with homologies in $\mathcal{A}$. Then we have :*

$$Ext^i(Z_r, A) \cong \begin{cases} Ext^d(H_{r+i-(d-1)}, A) & 1 \leq i \leq d-2 \\ 0 & for\ i \geq d-1 \end{cases} \tag{1}$$

$$Ext^i(B_r, A) \cong \begin{cases} Ext^d(H_{r+i-(d-1)}, A) & 1 \leq i \leq d-1 \\ 0 & for\ i \geq d \end{cases} \tag{2}$$

$$Ext^i\left(\frac{P_r}{B_r}, A\right) \cong \begin{cases} Ext^d(H_{r+i-d}, A) & 1 \leq i \leq d \\ 0 & i \geq d \end{cases} \tag{3}$$

**Proof.** Since $P_i = 0 \ \forall i \ll 0$, the theorem is true for $r \ll 0$. So, we assume that the theorem is true for $r-1$ and prove it for $r$.

Corresponding to the short exact sequence $0 \to Z_r \to P_r \to B_{r-1} \to 0$, we get a long exact Ext-sequence which yields

$$0 \to Ext^0(B_{r-1}, A) \to P_r^* \to Ext^0(Z_r, A) \to Ext^1(B_{r-1}, A) \to 0 \tag{4}$$

and for $i \geq 1$ we have $Ext^i(Z_r, A) \cong Ext^{i+1}(B_{r-1}, A)$. Thus, the induction hypothesis yields that for $i \geq 1$,

$$Ext^i(Z_r, A) = Ext^{i+1}(B_{r-1}, A) = \begin{cases} Ext^d(H_{r+i-(d-1)}, A) & 1 \leq i \leq d-2 \\ 0 & for\ i \geq d-1 \end{cases}$$

So, equation (1) is established.

Consider the long exact Ext-sequence corresponding to the short exact sequence $0 \to B_r \to Z_r \to H_r \to 0$. By (2.5), $Ext^i(H_r, A) = 0$ for all $i \neq d$ and since $Ext^i(Z_r, A) = 0 \ \ \forall i > d-2$ from equation (1), it follows that

$$\begin{aligned} Ext^i(B_r, A) &\cong \begin{cases} Ext^i(Z_r, A), & 0 \leq i \leq d-2 \\ Ext^d(H_r, A), & i = d-1 \\ 0, & i \geq d \end{cases} \\ &\cong \begin{cases} Ext^0(Z_r, A), & i = 0 \\ Ext^d(H_{r+i-(d-1)}, A), & 1 \leq i \leq d-1 \\ 0, & i \geq d \end{cases} \end{aligned}$$

Now consider the short exact sequence $0 \to H_r \to \frac{P_r}{B_r} \to B_{r-1} \to 0$. Again, $Ext^i(H_r, A) = 0 \ \forall \ i \neq d$ from (2.5) and from equation (2) we get $Ext^i(B_r, A) = 0 \ \forall i > d-1$. So it follows that

$$\begin{aligned} Ext^i\left(\frac{P_r}{B_r}, A\right) &\cong \begin{cases} Ext^i\left(B_{r-1}, A\right), & 0 \leq i \leq d-1 \\ Ext^d(H_r, A), & i = d \\ 0, & i > d \end{cases} \\ &\cong \begin{cases} Ext^i\left(B_{r-1}, A\right), & i = 0 \\ Ext^d(H_{r+i-d}, A), & 1 \leq i \leq d \\ 0, & i > d \end{cases} \end{aligned}$$

□

**Corollary 3.2.** *Suppose $P_\bullet$ is a complex in $Ch^b(\mathcal{P}(A))$ with homologies in $\mathcal{A}$. Then, for all $r \in \mathbb{Z}$*

$$Ext^i(B_r, A), \quad Ext^i(Z_r, A), \quad Ext^i\left(\frac{P_r}{B_r}, A\right)$$

*are in $\mathcal{A}$ for $i \geq 1$ and are in $\mathcal{MFPD}(A)$ for $i = 0$.*

**Proof.** By (2.5) and the preceding theorem (3.1), for $i \geq 1$, the statement is clear. For $i = 0$, we recall below equation (4) from the preceding proof :

$$0 \to Ext^0(B_{r-1}, A) \to P_r^* \to Ext^0(Z_r, A) \to Ext^1(B_{r-1}, A) \to 0.$$

and that we also proved $Ext^0(B_r, A) \cong Ext^0(Z_r, A)$ and $Ext^0\left(\frac{P_r}{B_r}, A\right) \cong Ext^0\left(B_{r-1}, A\right)$. Hence, it is enough to know that $Ext^0\left(B_{r-1}, A\right)$ satisfies the theorem. Once again induction saves the day! □

This allows us to conclude our main theorem of the section.

**Theorem 3.3.** *Let $P_\bullet$ be a complex as in theorem (3.1). Then, for $t \in \mathbb{Z}$, we have*

$$H_{-t}(P_\bullet^*) \cong Ext^d\left(H_{t-d}(P_\bullet), A\right) \cong H_{t-d}(P_\bullet)^\vee.$$

*In particular, $H_r(P_\bullet^*) \in \mathcal{A}$ and hence, $Ch^b_{\mathcal{A}}(\mathcal{P}(A))$ is closed under duality.*

**Proof.** Consider the dual complex : $\cdots P_{t-1}^* \stackrel{(\partial_t)^*}{\to} P_t^* \stackrel{(\partial_{t+1})^*}{\to} P_{t+1}^* \cdots$. Note that $(\partial_{t+1})^* : P_t^* \to P_{t+1}^*$ factors through

$$P_t^* \to B_t^* \hookrightarrow \left(\frac{P_{t+1}}{B_{t+1}}\right)^* \hookrightarrow P_{t+1}^*$$

(recall $d \geq 2$) and hence, $ker((\partial_{t+1})^*) = ker(P_t^* \to B_t^*) = \left(\frac{P_t}{B_t}\right)^*$. Similarly, $ker((\partial_t)^*) = \left(\frac{P_{t-1}}{B_{t-1}}\right)^*$ and hence we obtain the exact sequence

$$0 \to \left(\frac{P_{t-1}}{B_{t-1}}\right)^* \to (P_{t-1})^* \to \left(\frac{P_t}{B_t}\right)^* \to H_{-t}(P_\bullet^*) \to 0.$$

Note also that there is an exact Ext-sequence

$$0 \to \left(\frac{P_{t-1}}{B_{t-1}}\right)^* \to (P_{t-1})^* \to (B_{t-1})^* \to Ext^1\left(\frac{P_{t-1}}{B_{t-1}}, A\right) \to 0.$$

But since $d \geq 2$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left(\frac{P_{t-1}}{B_{t-1}}\right)^* & \longrightarrow & (P_{t-1})^* & \longrightarrow & (B_{t-1})^* & \longrightarrow & Ext^1\left(\frac{P_{t-1}}{B_{t-1}}, A\right) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow \wr & & \downarrow \wr & & \\
0 & \longrightarrow & \left(\frac{P_{t-1}}{B_{t-1}}\right)^* & \longrightarrow & (P_{t-1})^* & \longrightarrow & \left(\frac{P_t}{B_t}\right)^* & \longrightarrow & H_{-t}(P_\bullet^*) & \longrightarrow & 0
\end{array}$$

Hence, by (3.1), we get that $H_{-t}(P_\bullet^*) \cong Ext^d\left(H_{t-d}(P_\bullet), A\right)$. The rest follows from (2.5). $\square$

*Remark* 3.4. It is a straightforward diagram check that all the isomorphisms in (3.1) and (3.3) are natural. In particular, that means that if we have a morphism of complexes $P_\bullet \xrightarrow{f} Q_\bullet$, then there is a commutative diagram :

$$\begin{array}{ccc}
H_{-t}(Q_\bullet^*) & \xrightarrow[\sim]{H_{-t}(f^*)} & H_{-t}(P_\bullet^*) \\
\downarrow \wr & & \downarrow \wr \\
H_{t-d}(Q_\bullet)^\vee & \xrightarrow[H_{t-d}(f)^\vee]{\sim} & H_{t-d}(P_\bullet)^\vee
\end{array}$$

Finally, as an easy consequence of the above theorem, we obtain (for free!) that $D^b_{\mathcal{A}}(\mathcal{A})$ is closed under duality $M^\vee = Ext^d(M, A)$.

**Theorem 3.5.** *The category $Ch^b_{\mathcal{A}}(\mathcal{A})$ is closed under the duality $^\vee$ induced by the duality $^\vee$ in $\mathcal{A}$.*

**Proof.** Suppose $M_\bullet$ is a complex in $Ch^b_{\mathcal{A}}(\mathcal{A})$. Without loss of generality, we assume $M_\bullet$ is supported on $[n, 0]$. Each component $M_i$ has a projective resolution of length $d$, and putting them together with the induced maps, we get a double complex $L_{\bullet\bullet}$, as in the left figure below :

$$\begin{array}{ccccccccccccccc}
 & 0 & \to & \vdots & \to & \vdots & \to & \cdots & \to & \vdots & \to & \vdots & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
L_{\bullet\bullet} = & 0 & \to & L_{1n} & \to & L_{1(n-1)} & \to & \cdots & \to & L_{11} & \to & L_{10} & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & 0 & \to & L_{0n} & \to & L_{0(n-1)} & \to & \cdots & \to & L_{01} & \to & L_{00} & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
M_\bullet = & 0 & \to & M_n & \to & M_{n-1} & \to & \cdots & \to & M_1 & \to & M_0 & \to & 0
\end{array}
\qquad
\begin{array}{ccccccccccccccc}
 & 0 & \to & \vdots & \to & \vdots & \to & \cdots & \to & \vdots & \to & \vdots & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
L'_{\bullet\bullet} = & 0 & \to & L^*_{(d-1)0} & \to & L^*_{(d-1)1} & \to & \cdots & \to & L^*_{(d-1)(n-1)} & \to & L^*_{(d-1)n} & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & L^*_{d0} & \to & L^*_{d1} & \to & \cdots & \to & L^*_{d(n-1)} & \to & L^*_{dn} & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_\bullet^\vee = & 0 & \to & M_0^\vee & \to & M_1^\vee & \to & \cdots & \to & M_{n-1}^\vee & \to & M_n^\vee & \to & 0
\end{array}$$

Dualizing, $L'_{\bullet\bullet}$ gives a similar resolution of $M_\bullet^\vee$, as shown on the right above (note that there are sign conventions on the differentials of the complexes here, in particular $M_\bullet^\vee$ acquires a $(-1)^d$ factor on its differentials. We refer to [W] for the conventions for total complexes.)
Now, the total complexes give quasi-isomorphisms

$$Tot(L_{\bullet\bullet}) \longrightarrow M_\bullet, \quad Tot(L'_{\bullet\bullet}) \twoheadrightarrow M_\bullet^\vee.$$

So, $H_i(Tot(L_{\bullet\bullet})) \in \mathcal{A}$ for all $i \in \mathbb{Z}$. By (3.3), $H_i(Tot(L^*_{\bullet\bullet})) \in \mathcal{A}$ for all $i \in \mathbb{Z}$. Now after translating $Tot(L_{\bullet\bullet})^*$ $d$ components to the left, we observe that it is actually chain homotopy equivalent to $Tot(L'_{\bullet\bullet})$ and so we have $H_i(Tot(L'_{\bullet\bullet})) \in \mathcal{A}$. Finally, the above quasi-isomorphism yields that $H_i(M_\bullet^\vee) \xrightarrow{\sim} H_i(Tot(L'_{\bullet\bullet})) \in \mathcal{A}$. This completes the proof. The proof is complete. □

# 4 Definitions of Witt Groups

In this section, we define Witt groups of the categories we work with. In particular, we extend the definition of Witt groups from triangulated categories with duality to their additive subcategories which are closed under orthogonal sums, translations and isomorphisms. Since it is possible that there is cause for confusion about translation, we start by clearing the air.

**Definition 4.1.** In all the categories of complexes, there are two possible translations, $T_u$ and $T_s$. The complex $T_uP_\bullet$ is defined as $(T_uP_\bullet)_i = P_{i-1}$ and $\partial(T_uP_\bullet)_i = \partial(P_\bullet)_{i-1}$. The complex $T_sP_\bullet$ is defined as $(T_sP_\bullet)_i = P_{i-1}$ and $\partial(T_sP_\bullet)_i = -\partial(P_\bullet)_{i-1}$.

**Note that $T_s$ seems to be the "standard" translation in literature and that is always the translation we use on any category of complexes.**

However, given a duality $*$ on such a category (e.g. $D^b_{\mathcal{A}}(\mathcal{A})$ and $D^b_{\mathcal{A}}(\mathcal{P}(A))$), there are shifted dualities, $T_s^n{\circ}*$ and $T_u^n{\circ}*$. We work with the unsigned duality $T_u^n{\circ}*$ until we reach section 6. Note however that $H_i(T_s^nP_\bullet^*) = H_i(T_u^nP_\bullet^*)$ and so much of what we will say is independent of the chosen duality.

*Remark* 4.2. We quickly review the situation for the categories $\mathcal{A}$ and $\mathcal{R}$. First note that both of these categories are exact categories with duality and so the Witt groups are defined as in [QSS].

The functor $\iota$ induces duality preserving equivalences

$$\iota : (\mathcal{A},^\vee, \tilde{\varpi}) \longrightarrow \left(\mathcal{R}(A), T_u^d{\circ}*, \varpi\right), \quad \iota : (\mathcal{A},^\vee, -\tilde{\varpi}) \longrightarrow \left(\mathcal{R}(A), T_u^d{\circ}*, -\varpi\right)$$

of categories which then yield isomorphisms of the corresponding Witt groups

$$W(\iota) : W\left(\mathcal{A},^\vee, \tilde{\varpi}\right) \xrightarrow{\sim} W\left(\mathcal{R}(A), T_u^d{\circ}*, \varpi\right),$$

$$W(\iota) : W\left(\mathcal{A},^{\vee}, -\tilde{\varpi}\right) \xrightarrow{\sim} W\left(\mathcal{R}(A), T_u^d \circ *, -\varpi\right)$$

Finally, we get to our definitions of the Witt group. Given an exact category $\mathcal{E}$, its derived category will be denoted by $D^b(\mathcal{E})$. For a subcategory $\mathcal{C}$, $D^b_{\mathcal{C}}(\mathcal{E})$ will denote the full subcategory of $D^b(\mathcal{E})$ consisting of complexes with homologies in $\mathcal{C}$.

The derived categories which will be used in this article include :

$$D^b_{\mathcal{A}}(\mathcal{P}(A)) \subseteq D^b_{fl}(\mathcal{P}(A)) \subseteq D^b(\mathcal{P}(A)) \qquad\qquad D^b_{\mathcal{A}}(\mathcal{A}) \subseteq D^b(\mathcal{A}).$$

We have the following diagram of subcategories and functors:

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mu} & D^b_{\mathcal{A}}(\mathcal{A}) & \xhookrightarrow{\nu} & D^b(\mathcal{A}) \\ {\scriptstyle\iota}\downarrow & \searrow^{\zeta} & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \mathcal{R}(A) & \xhookrightarrow{\mu'} & D^b_{\mathcal{A}}(\mathcal{P}(A)) & \xhookrightarrow{\nu'} & D^b_{fl}(\mathcal{P}(A)). \end{array} \tag{5}$$

Here $\mu(M)$ is the complex concentrated at degree zero. The functor $\iota(M)$ is obtained by making a choice of projective resolution of length $d$ and then defining $M^{\vee} = H_0((\zeta(M))^*)$. The functors $\alpha$ and $\beta$ are essentially induced by these ones, by taking the total complex (take a look at the proof of (3.5)).

We now move on towards the definitions of the Witt groups of the categories $D^b_{\mathcal{A}}(\mathcal{A})$ and $D^b_{\mathcal{A}}(\mathcal{P}(A))$. We once again remind the reader that this definition relies on the definitions in [TWGI].

**Definition 4.3.** Let $\delta = \pm 1$. Suppose $K := (K, \#, \delta, \varpi)$ is a triangulated category with translation $T$ and $\delta-$duality $\#$. Suppose $K_0$ is a full subcategory of $K$ that is closed under isomorphism, translation and orthogonal sum. We abuse notation and denote $K_0 := (K_0, \#, \delta, \varpi)$ in order to keep track of the duality and canonical isomorphism in use.

1. Define the Witt monoid of $MW(K_0)$ to be the submonoid

$$MW(K_0) = \{(P, \varphi) \in MW(K) : P \in Ob(K_0)\}.$$

2. A symmetric space $(P, \varphi) \in MW(K_0)$ will be called a **neutral space** in $MW(K_0)$ if it has a Lagrangian $(L, \alpha, w)$ in $MW(K)$ such that $L, L^{\#} \in Ob(K_0)$.

3. Let $NW(K_0)$ be the submonoid of $MW(K_0)$ generated by the isometry classes of neutral spaces in $K_0$.

4. Define the Witt group

$$W(K_0) := \frac{MW(K_0)}{NW(K_0)}.$$

   Note $(Q, \chi) \in MW(K_0) \Longrightarrow (Q, -\chi) \in MW(K_0)$. It is easy to check that $(Q, \chi) \perp (Q, -\chi) \in NW(K_0)$. So, $W(K_0)$ has a group structure. We use this definition in the context of derived categories of exact categories with duality.

5. Let $(\mathcal{C}, ^\vee, \varpi)$ be an exact subcategory with duality in an ambient abelian category $\mathcal{C}'$ and let $\mathcal{D}$ be any subcategory of $\mathcal{C}'$ closed under orthogonal sum. Let $K_0 = D^b_{\mathcal{D}}(\mathcal{C})$ (2.1). Then with the induced duality and natural isomorphism, the Witt group $W\left(D^b_{\mathcal{D}}(\mathcal{C}), ^\vee, \delta, \varpi\right)$ is defined as above.

6. Accordingly, with $T = T_u, T_s$, the Witt groups

$$W\left(D^b_{\mathcal{A}}\left(\mathcal{P}(A)\right), T^n{}_\circ *, \pm 1, \pm\varpi\right), \quad W\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right), T^n{}_\circ *, \pm 1, \pm\tilde{\varpi}\right)$$

   are defined.

# 5 Isomorphisms of Witt Groups

All the functors above induce homomorphisms of Witt groups. As always, $A$ **denotes a Cohen-Macaulay ring with** $\dim A_m = d \geq 2$**, for all maximal ideals** $m$ **of** $A$. Let $D^b_{\mathcal{A}}\left(\mathcal{A}, ^\vee, \pm\tilde{\varpi}\right)$, $D^b_{\mathcal{A}}\left(\mathcal{A}, ^\vee, \pm\tilde{\varpi}\right)$ denote the dulity structure, respectively, on $D^b_{\mathcal{A}}(\mathcal{A})$ and $D^b(\mathcal{A})$ induced by $(\mathcal{A},^{\vee}, \pm\tilde{\varpi})$ and

$$D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u := \left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T^d_{u^\circ}*, 1, \pm\varpi\right).$$

Recall the functors in the diagram (5), it is clear that the functors $\mu, \nu, \mu'$ and $\gamma$ preserve dualities. For $\iota$, this is left to the reader as a diagram chase (but note the definition of $M^{\vee}$ after that same diagram). Essentially the same proof also gives us that $\zeta, \alpha$ and $\beta$ are duality preserving. This being done, we can talk about the corresponding maps of the Witt groups. The goal of this section is to establish the following diagram of homomorphisms of Witt groups :

$$\begin{array}{ccccc}
W(\mathcal{A},^{\vee}, \pm\tilde{\varpi}) & \xrightarrow[\sim]{W(\mu)} & W\left(D^b_{\mathcal{A}}(\mathcal{A},^{\vee}, \pm\tilde{\varpi})\right) & \xrightarrow[\sim]{W(\nu)} & W\left(D^b(\mathcal{A},^{\vee}, \pm\tilde{\varpi})\right) \\
{\scriptstyle W(\iota)}\downarrow\wr & \searrow^{\sim}_{W(\zeta)} & {\scriptstyle W(\alpha)}\downarrow\wr & & \\
W\left(\mathcal{R}(A), T^d_u*, \pm\varpi\right) & \xrightarrow[\sim]{W(\gamma)} & W\left(D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u\right) & &
\end{array} \tag{6}$$

Note that we already know that $W(\iota)$ is an isomorphism (4.2) and further, by [TWGII, Theorem 4.3], $W(\nu\circ\mu)$ is an isomorphism. The proof that

$$W(\mu) : W(\mathcal{A},^{\vee}, \pm\tilde{\varpi}) \longrightarrow W\left(D^b_{\mathcal{A}}(\mathcal{A},^{\vee}, \pm\tilde{\varpi})\right)$$

are isomorphisms follows from more abstract "general nonsense" which we prove in (A.1) as part of the appendix (A). Since $W(\nu\circ\mu)$ and $W(\mu)$ are isomorphisms, it is clear that so is $W(\nu)$. The main result of this section is that $W(\zeta)$ is an isomorphism. That being established, it is clear that $W(\gamma)$ and $W(\alpha)$ are isomorphisms.

For the rest of this section, we use the notation $\# := T^d_{u^\circ}*$. First, we establish the following regarding the structure of symmetric forms.

**Lemma 5.1.** *Suppose* $\eta : X_\bullet \xrightarrow{\sim} X^{\#}_\bullet$ *is a symmetric form in* $D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u$*, such that*

$$H_{-m}(X_\bullet) \neq 0, \quad \textit{and} \quad H_i(X_\bullet) = 0 \quad \textit{for all} \quad i < -m.$$

*Then, there is a complex* $P_\bullet$ *in* $Ch^b_{\mathcal{A}}(\mathcal{P}(A))$ *and a quasi-isomorphism* $\varphi : P_\bullet \to P^{\#}_\bullet$ *such that*

1. $(P_\bullet, \varphi)$ *is isometric to* $(X_\bullet, \eta)$ *in* $D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u$.

2. $P_\bullet$ *is supported on* $[m+d, -m]$.

3. $H_{-m}(P_\bullet) \neq 0$.

**Proof.** Recall from (2.9) that since $H_{-m}(X_\bullet) \neq 0$, $X_\bullet$ has length at least $d$. By duality, we conclude that $m \geq 0$. By definition there is a complex $P_\bullet$ of projective modules and a quasi-isomorphism $t : P_\bullet \longrightarrow X_\bullet$, a chain complex morphism $\varphi_0 : P_\bullet \longrightarrow X_\bullet^\#$ such that $\eta = \varphi_0 t^{-1}$. Then, $\varphi = t^\# \varphi_0 = t^\# \eta t$ is a symmetric form on $P_\bullet$, and $(X_\bullet, \eta)$ is isometric to $(P_\bullet, \varphi)$. By including enough zeros on the two tails, we can assume $P_\bullet$ is supported on $[n+d, -n]$, for some $n \geq m$. If $m = n$ there is nothing to prove. So, assume $n > m$. We have, $H_{-n}(P_\bullet) = 0$. Inductively, we will cut down the support to $[m+d, m]$. We write $\varphi : P_\bullet \longrightarrow P_\bullet^\#$ as follows

$$
\begin{array}{ccccccccccccc}
\longrightarrow & 0 & \longrightarrow & P_{n+d} & \xrightarrow{\partial_{n+d}} & P_{(n-1)+d} & \xrightarrow{\partial_{(n-1)+d}} & \cdots & \xrightarrow{\partial_{-(n-2)}} & P_{-(n-1)} & \xrightarrow{\partial_{-(n-1)}} & P_{-n} & \longrightarrow 0 \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{(n-1)+d}} & & & & \downarrow{\scriptstyle \varphi_{-(n-1)}} & & \downarrow{\scriptstyle \varphi_{-n}} & \quad\downarrow \\
\longrightarrow & 0 & \longrightarrow & P^*_{-n} & \xrightarrow[\partial^*_{-(n-1)}]{} & P^*_{-(n-1)} & \xrightarrow{\partial^*_{-(n-2)}} & \cdots & \xrightarrow[\partial^*_{(n-1)+d}]{} & P^*_{(n-1)+d} & \xrightarrow[\partial^*_{n+d}]{} & P^*_{n+d} & \longrightarrow 0 \longrightarrow
\end{array}
$$

where $P_i$ are finitely generated projective $A-$modules. Since $n > m, H_{-n}(P_\bullet) \cong H_{-n}(P_\bullet^*) \cong 0$. So, $\partial_{-(n-1)}$ and $\partial^*_{n+d}$ are both split surjections. Thus there are homomorphisms $\epsilon_{-n} : P_{-n} \longrightarrow P_{-(n-1)}$ and $\epsilon^*_{n+d} : P_{n+d}* \longrightarrow P_{n-1+d}$ such that $\partial_{-(n-1)} \circ \epsilon_{-n} = Id$ and $\partial^*_{n+d} \circ \epsilon^*_{n+d} = Id$. Hence, $Z_{-(n-1)}$ and $\frac{P_{n-1+d}}{P_{n+d}}$ are projective modules. Note that since $d \geq 2$, by (2.9), $\frac{P_{n-1+d}}{P_{n+d}} = B_{n-2+d}$. Further, we obtain splittings $\sigma_{-(n-1)} : P_{-(n-1)} \twoheadrightarrow Z_{-(n-1)}$ and $\sigma_{(n-2)+d} : B_{(n-2)+d} \hookrightarrow P_{(n-1)+d}$. This gives us a shorter complex $Q_\bullet$, naturally chain homotopic to $P_\bullet$ and an induced symmetric form on $Q_\bullet$ :

$$
\begin{array}{ccccccccccccccc}
Q_\bullet : & \longrightarrow 0 & \longrightarrow & 0 & \longrightarrow & B_{(n-2)+d} & \hookrightarrow & P_{(n-2)+d} & \cdots \xrightarrow{\partial} & Z_{-(n-1)} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
\downarrow & \downarrow & & \downarrow & & \downarrow{\scriptstyle \sigma_{(n-2)+d}} & & \| & & \downarrow & & \downarrow & & \downarrow & \\
P_\bullet : & \longrightarrow 0 & \longrightarrow & P_{n+d} & \xrightarrow{\partial_{n+d}} & P_{(n-1)+d} & \xrightarrow{\partial} & P_{(n-2)+d} & \cdots \xrightarrow{\partial} & P_{-(n-1)} & \xrightarrow{\partial} & P_{-n} & \longrightarrow & 0 & \longrightarrow \\
{\scriptstyle \varphi}\downarrow & \downarrow & & \downarrow{\scriptstyle \varphi_{n+d}} & & \downarrow{\scriptstyle \varphi_{(n-1)+d}} & & \downarrow{\scriptstyle \varphi_{(n-2)+d}} & & \downarrow{\scriptstyle \varphi_{-(n-1)}} & & \downarrow{\scriptstyle \varphi_{-n}} & & \downarrow & \\
P_\bullet^\# : & \longrightarrow 0 & \longrightarrow & P^*_{-n} & \xrightarrow[\partial^*_{-(n-1)}]{} & P^*_{-(n-1)} & \xrightarrow[\partial^*_{-(n-2)}]{} & P^*_{-(n-2)} & \cdots \xrightarrow[\partial^*]{} & P^*_{(n-1)+d} & \xrightarrow[\partial^*_{n+d}]{} & P^*_{n+d} & \longrightarrow & 0 & \longrightarrow \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle \sigma^*_{(n-2)+d}} & & \downarrow & & \downarrow & \\
Q_\bullet^\# : & \longrightarrow 0 & \longrightarrow & 0 & \longrightarrow & Z^*_{-(n-1)} & \longrightarrow & P^*_{-(n-2)} & \cdots \xrightarrow[\partial^*]{} & B^*_{(n-2)+d} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow
\end{array}
$$

Calling this map $\varphi'$, $(Q_\bullet, \varphi')$ is obviously isometric to $(P_\bullet, \varphi)$ and hence to the original form $(X_\bullet, \eta)$. Since $Q_\bullet$ is supported on $[(n-1)+d, -(n-1)]$ induction finishes the proof. $\square$

Since $\zeta$ is given by composing $\nu$ and $\alpha$ and there are maps $W(\nu)$ and $W(\alpha)$, it is clear that $W(\zeta)$ is well-defined. However, we give an explicit proof which might also be somewhat illuminating considering the unsaid details about why duality-preserving functors induce maps of Witt groups. The proof essentially follows the proof in [DWG, 2.11].

**Theorem 5.2.** *The functor $\zeta$ induces a well defined homomorphism*

$$W(\zeta) : W\left(\mathcal{A},^\vee, \pm\tilde{\varpi}\right) \longrightarrow W\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{\pm}_u\right).$$

**Proof.** We will only prove

$$W(\zeta) : W\left(\mathcal{A},^\vee, \tilde{\varpi}\right) \longrightarrow W\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{+}_u\right).$$

is well defined and the case of skew dualities follows similarly. It is clear that $\zeta$ defines a well-defined map from $MW\left(\mathcal{A},^\vee, \tilde{\varpi}\right)$ to $MW\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{+}_u\right)$ since projective maps of modules can be lifted to a chain complex map of their resolutions (note that though the lift is not unique, it is unique upto homotopy and so gives the same morphism in $D^b_{\mathcal{A}}(\mathcal{P}(A))$). So we need to check that the image of a neutral space in $MW\left(\mathcal{A},^\vee, \tilde{\varpi}\right)$ is neutral in $MW\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{+}_u\right)$.

Suppose $(M, \varphi_0)$ is a neutral space in $(\mathcal{A},^\vee, \tilde{\varpi})$. Let $\alpha_0 : N \longrightarrow M$ be a lagrangian of $(M, \varphi_0)$. Then

$$0 \longrightarrow N \xrightarrow{\alpha_0} M \xrightarrow{\alpha_0^\vee \varphi_0} N^\vee \longrightarrow 0 \quad \textit{is exact.}$$

Suppose $L_\bullet, P_\bullet$ are the chosen projective resolutions of $N$ and $M$ and $\alpha : L_\bullet \to P_\bullet$ is the morphism induced from $\alpha_0$. The above short exact sequence implies the composition $L_\bullet \overset{\alpha}{\to} P_\bullet \overset{\alpha^\# \varphi}{\to} L^\#_\bullet$ is chain homotopic to 0 (hence the 0 map in $D^b_{\mathcal{A}}(\mathcal{P}(A))$). Completing $\alpha$ to an exact triangle, we get a morphism of exact triangles

$$\begin{array}{ccccccc}
L_\bullet & \xrightarrow{\alpha} & P_\bullet & \xrightarrow{j} & C_\bullet & \xrightarrow{k} & T(L_\bullet) \\
\downarrow & & \downarrow{\scriptstyle \alpha^\# \varphi} & & \downarrow{\scriptstyle s} & & \downarrow \\
0 & \longrightarrow & L^\#_\bullet & = & L^\#_\bullet & \longrightarrow & 0
\end{array}$$

Note that $H_0(C_\bullet) \cong N^\vee$ and $\forall \;\; i \neq 0 \quad H_i(C_\bullet) = 0$ and so $C_\bullet$ is an object in $D^b_{\mathcal{A}}(\mathcal{P}(A))$. The map $s$ is actually quite easy to describe, namely $s = (0, \alpha^\# \varphi) : L_{n-1} \oplus P_n \longrightarrow L_n^*$ and it follows from the above morphism of triangles (or by direct checking) that $s$ is a quasi-isomorphism. Hence,

$$L_\bullet \xrightarrow{\alpha} P_\bullet \xrightarrow{\alpha^\# \varphi} L_\bullet^\# \xrightarrow{k \circ s^{-1}} T(L_\bullet)$$

is an exact triangle. Setting $w = -T^{-1}(k \circ s^{-1})$, we get an exact triangle

$$T^{-1}(L_\bullet^\#) \xrightarrow{w} L_\bullet \xrightarrow{\alpha} P_\bullet \xrightarrow{\alpha^\# \varphi} L_\bullet^\#$$

Now all we require is that $T^{-1}w^\# = w$.

$$\begin{aligned} T^{-1}w^\# = w \Leftrightarrow T^{-1}w^\# = -T^{-1}(k \circ s^{-1}) \Leftrightarrow (T^{-1}(k \circ s^{-1}))^\# = k \circ s^{-1} \\ \Leftrightarrow T(s^{-1\#} \circ k^\#) = k \circ s^{-1} \Leftrightarrow T(k^\#) \circ s = T(s^\#) \circ k. \end{aligned}$$

A quick physical check of the maps in question yields that the first map is

$$L_{n-1} \oplus P_n \xrightarrow{\begin{pmatrix} -1 & 0 \end{pmatrix}} L_{n-1} \xrightarrow{\begin{pmatrix} 0 \\ \varphi^*_{n-1} \circ \alpha_{d-n+1} \end{pmatrix}} L^*_{d-n} \oplus P^*_{d-n+1}$$

while the second one is

$$L_{n-1} \oplus P_n \xrightarrow{\begin{pmatrix} 0 & \alpha^*_{d-n} \circ \varphi_n \end{pmatrix}} L^*_{d-n} \xrightarrow{\begin{pmatrix} -1 \\ 0 \end{pmatrix}} L^*_{d-n} \oplus P^*_{d-n+1}$$

The matrices we thus obtain are

$$\begin{pmatrix} 0 & -\varphi^*_{n-1} \circ \alpha_{d-n+1} \\ 0 & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & 0 \\ \alpha^*_{d-n} \circ \varphi_n & 0 \end{pmatrix}$$

which are homotopy equivalent using

$$\tau = \begin{pmatrix} 0 & 0 \\ 0 & (-1)^n \varphi \end{pmatrix}.$$

Therefore, $(L_\bullet, \alpha, w)$ is a lagrangian. Hence $W(\zeta)$ is a well defined homomorphism of groups. $\square$

We now proceed towards (5.11) which proves that $W(\zeta)$ is surjective. The main tool here is to construct a special sublagrangian and then use Balmer's sublagrangian construction [TWGI, Section 4 and Theorem 4.20] to reduce the length of $(P, \varphi)$.

*Remark* 5.3. Note that using (5.1), any symmetric form $(X_\bullet, \phi)$ in $\left(D^b_{\mathcal{A}}(\mathcal{A})^+_u\right)$ with $X_\bullet$ not acyclic can be represented by

$$
\begin{array}{ccccccccccc}
P_\bullet = \cdots\ 0 & \longrightarrow & P_{n+d} & \xrightarrow{\partial} & P_{(n-1)+d} & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & P_{-(n-1)} & \xrightarrow{\partial} & P_{-n} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{(n-1)+d}} & & \downarrow{\scriptstyle \varphi_{-(n-1)}} & & \downarrow{\scriptstyle \varphi_{-n}} & & \\
P^{\#}_\bullet = \cdots\ 0 & \longrightarrow & P^*_{-n} & \xrightarrow[\partial^*]{} & P^*_{-(n-1)} & \longrightarrow \cdots \xrightarrow[\partial^*]{} & P^*_{(n-1)+d} & \xrightarrow[\partial^*]{} & P^*_{n+d} & \longrightarrow & 0
\end{array}
$$

with $H_{-n}(P_\bullet) \neq 0$.

**Lemma 5.4.** *Let $(P_\bullet, \varphi)$ be as above. Then*

1. $H_r(P_\bullet) = 0$ *for* $r = n+1, n+2, \ldots, n+d$.
2. $H_n(P_\bullet) \neq 0$.

**Proof.** The first point follows from (2.9). To prove (2), assume $H_n(P_\bullet) = 0$. Then, with $B_{n-1} = image(\partial_n)$ we have an exact sequence

$$0 \longrightarrow P_{n+d} \longrightarrow P_{(n-1)+d} \longrightarrow \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \tfrac{P_{n-1}}{B_{n-1}} \longrightarrow 0$$

Since this is a projective resolution of the last term, if follows

$$H_{-n}(P^*_\bullet) = Ext^{d+1}\left(\frac{P_{n-1}}{B_{n-1}}, A\right) = 0.$$

This is a contradiction to $H_{-n}(P_\bullet) \neq 0$. The proof is complete. $\square$

Much of what follows is dependent on [TWGI, §4] and the interested reader is highly encouraged to take a look at it. We recall the definition of a sublagrangian of $(P_\bullet, \varphi)$ :

**Definition 5.5.** A sublagrangian of a symmetric form $(P_\bullet, \varphi)$ is a pair $(L_\bullet, \alpha)$ with $L_\bullet \in Ob(D^b_{\mathcal{A}}(\mathcal{P}(A)))$ and $\alpha : L_\bullet \to P_\bullet$, which satisfies that $\alpha^{\#} \circ \varphi \circ \alpha = 0$ in $D^b_{\mathcal{A}}(\mathcal{P}(A))$.

For $(P_\bullet, \varphi)$ as above, (5.4) tells us that $H_n(P_\bullet) \neq 0$ and we already know it is in $\mathcal{A}$. So it has a minimal projective resolution of length $d$. Let $L_\bullet$ be a projective resolution of $H_n(P_\bullet)$ of length $d$, shifted by $n$ places, as in the diagram below. Since $H_i(P_\bullet)\ \forall\ i > n$ by (2.9), the bottom line is a projective resolution of $\frac{P_n}{B_n}$ and so the inclusion $H_n(P_\bullet) \hookrightarrow \frac{P_n}{B_n}$ induces a map of complexes

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & L_{n+d} & \longrightarrow & L_{(n-1)+d} & \longrightarrow & \cdots & \longrightarrow & L_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nu_{n+d}} & & \downarrow{\scriptstyle \nu_{(n-1)+d}} & & & & \downarrow{\scriptstyle \nu_n} & & \\
0 & \longrightarrow & P_{n+d} & \longrightarrow & P_{(n-1)+d} & \longrightarrow & \cdots & \longrightarrow & P_n & \longrightarrow & 0.
\end{array}$$

Note that since the composition $H_n(P_\bullet) \hookrightarrow \frac{P_n}{B_n} \twoheadrightarrow B_{n-1}$ is 0, we get a chain complex morphism $\nu : L_\bullet \longrightarrow P_\bullet$.

**Lemma 5.6.** *With the notations as above, for $n > 0, (L_\bullet, \nu)$ defines a sub-lagrangian of $(P_\bullet, \varphi)$.*

**Proof.** Let $\alpha = \nu^{\#}\varphi\nu$ is as follows (*the first line indicates the degrees*):

$$\begin{array}{cccccccccccccccc}
 & & & & n+1 & & n & & n-1 & & & & -n & & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & \\
\longrightarrow & L_{n+2} & \xrightarrow{\partial} & & L_{n+1} & \xrightarrow{\partial} & L_n & \longrightarrow & 0 & \longrightarrow & \cdots & & 0 & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow{\scriptstyle \alpha_{n+2}} & & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow & & & & \downarrow & & & \\
\longrightarrow & L^*_{d-(n+2)} & \xrightarrow[\partial^*_{d-n-1}]{} & & L^*_{d-(n+1)} & \xrightarrow[\partial^*_{d-n}]{} & L^*_{d-n} & \xrightarrow[\partial^*_{d-n+1}]{} & L^*_{d-n+1} & \xrightarrow[\partial^*]{} & \cdots & \longrightarrow & L^*_{n+d} & \longrightarrow & 0 & \longrightarrow
\end{array}$$

$L^{\#}$ is exact at all degrees except $-n$. Since $n > 0, H_i(L^{\#}) = 0 \forall i \geq n$. Hence, $image(\alpha_n) \subseteq ker(\partial^*_{d-n+1}) = image(\partial^*_{d-n})$. So, $\alpha_n$ lifts to a homomorphism $h_n : L_n \longrightarrow L^*_{d-(n+1)}$, i.e. ${\partial_{d-n}}^* h_n = \alpha_n$. So ${\partial_{d-n}}^*(\alpha_{n+1} - h_n\partial_{n+1}) = 0$. Now we can inductively define a homotopy $h_r : L_r \longrightarrow L^*_{d-(r+1)}$ so that $\alpha$ is homotopic to zero. The proof is complete. $\square$

We intend to apply the sulagrangian construction of Balmer [TWGI, Theorem 4.20] to $\nu$. Since $D^b_{\mathcal{A}}(\mathcal{P}(A))$ is not a triangulated category (in particular not closed under cones), we need to reprove some of the results in [TWGI, Theorem 4.20]. The main (and only) thing we have to keep track of is that in all the constructions, our objects remain within the category $D^b_{\mathcal{A}}(\mathcal{P}(A))$.

We start by checking that the cone of $\nu$ constructed in (5.6) is an object of $D^b_{\mathcal{A}}(\mathcal{P}(A))$.

**Lemma 5.7.** *With the notations of (5.6), let $N_\bullet$ be the cone of $\nu$. Then,*

1. *$N_\bullet$ is in $D^b_{\mathcal{A}}(\mathcal{P}(A))$.*
2. *The homologies are given by*

$$H_i(N_\bullet) = \begin{cases} H_i(P_\bullet) & if\ n > i \geq -n \\ 0 & otherwise \end{cases}$$

3. *$N_\bullet$ is supported on $[n+d+1, -n]$.*

**Proof.** The last point is obvious from the construction of the cone and (1) follows from (2). We prove (2). We have the exact triangle

$$T^{-1}N_\bullet \longrightarrow L_\bullet \xrightarrow{\nu} P_\bullet \longrightarrow N_\bullet.$$

By construction, $H_n(L_\bullet) \xrightarrow{\sim} H_n(P_\bullet)$ and $H_i(L_\bullet) = 0$ for all $i \neq n$. The long exact sequence of homologies

$$\begin{array}{l} \cdots H_{n+2}(N_\bullet) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_{n+1}(N_\bullet) \longrightarrow H_n(L_\bullet) \xrightarrow{\sim} H_n(P_\bullet) \longrightarrow \\ H_n(N_\bullet) \longrightarrow 0 \longrightarrow H_{n-1}(P_\bullet) \longrightarrow H_{n-1}(N_\bullet) \longrightarrow 0 \longrightarrow \cdots\cdots \longrightarrow \\ 0 \longrightarrow H_{-n}(P_\bullet) \longrightarrow H_{-n}(N_\bullet) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_{-(n+1)}(N_\bullet) \longrightarrow 0 \cdots \end{array}$$

establishes (2) and hence the lemma. □

Now we consider the dual $N^\#_\bullet$ of the cone of $\nu$.

**Lemma 5.8.** *With the same notations as above (in (5.6)), consider the following morphism of exact triangles :*

$$\begin{array}{ccccccc} T^{-1}N_\bullet & \xrightarrow{\nu_0} & L_\bullet & \xrightarrow{\nu} & P_\bullet & \xrightarrow{\nu_2} & N_\bullet \\ \downarrow{\scriptstyle T^{-1}\mu_0} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mu_0^\#} \\ T^{-1}L^\#_\bullet & \xrightarrow[T^{-1}\nu_0^\#]{} & N^\#_\bullet & \xrightarrow[\nu_2^\#]{} & P^\#_\bullet & \xrightarrow[\nu^\#]{} & L^\#_\bullet \end{array}$$

*(refer [TWGI, 4.3]...the existence of $\mu_0$ is assured by combining axioms (TR1) and (TR3) of triangulated categories and using that 2 is invertible.)*
*Then,*

1. *$N_\bullet^\#$ is in $D^b_{\mathcal{A}}(\mathcal{P}(A))$.*

2. *$N_\bullet^\#$ is supported on $[n+d, -(n+1)]$.*

3. *$\mu_0$ induces an isomorphism of the $n^{th}-$homology*

$$H(\mu_0) : H_n(L_\bullet) \xrightarrow{\sim} H_n(N_\bullet^\#).$$

4. $$H_i(N_\bullet^\#) \cong \begin{cases} H_i(P_\bullet^\#) & if\ n \geq i > -n \\ 0 & otherwise \end{cases}$$

**Proof.** (1) follows directly from (3.1). (2) follows because by (5.7) $N_\bullet$ is supported on $[(n+1)+d, -n]$. For (3), notice that the only nonzero homology of $L_\bullet^\#$ is at degree $-n$. Since $n > 0$, the long exact homology sequence of the second triangle gives us

$$H(\nu_2^\#) : H_n(N_\bullet^\#) \xrightarrow{\sim} H_n(P_\bullet^\#).$$

By choice of $\nu$ and $\varphi$, we know that

$$H(\nu) : H_n(L_\bullet) \xrightarrow{\sim} H_n(P_\bullet), \quad H(\varphi) : H_n(P_\bullet) \xrightarrow{\sim} H_n(P_\bullet^\#)$$

and hence, the commutative diagram

$$\begin{array}{ccc} H_n(L_\bullet) & \xrightarrow[\sim]{H_n(\nu)} & H_n(P_\bullet) \\ \downarrow{\scriptstyle H_n(\mu_0)} & & \wr\downarrow{\scriptstyle H_n(\varphi)} \\ H_n(N_\bullet^\#) & \xrightarrow[\sim]{H_n(\nu_2^\#)} & H_n(P_\bullet^\#) \end{array}$$

gives us (3). We prove (4) now. Since the only nonzero homology of $L_\bullet^\#$ is at degree $-n$, it is clear from the long exact homology sequence for the bottom exact triangle that

$$H_i(N_\bullet^\#) \cong H_i(P_\bullet^\#)\ \forall\ i \neq -n, -n-1.$$

By (5.7), $H_i(N_\bullet) = 0$ for all $i \geq n$ and so

$$H_{-(n+1)}(N^{\#}_\bullet) = Ext^{d+2}\left(\frac{N_{n-1}}{B_{n-1}}, A\right) = 0, \;\; H_{-n}(N^{\#}_\bullet) = Ext^{d+1}\left(\frac{N_{n-1}}{B_{n-1}}, A\right) = 0.$$

where $B_{n-1} \subseteq N_{n-1}$ is the boundary submodule (the last part also follows directly because $Ext^d(\frac{P_n}{B_n}, A) \cong Ext^d(H_n(P_\bullet), A))$. So, (4) is established. The proof is complete. $\square$

Now we consider the cone of $\mu_0$.

**Lemma 5.9.** *With the notations in (5.6), (5.7) and (5.8), consider an exact triangle on $\mu_0$ as follows:*

$$L_\bullet \xrightarrow{\mu_0} N^{\#}_\bullet \xrightarrow{\mu_1} R_\bullet \xrightarrow{\mu_2} T(L_\bullet)$$

*where $R_\bullet$ is the cone of $\mu_0$. Then $R_\bullet$ is an object of $D^b_{\mathcal{A}}(\mathcal{P}(A))$. More precisely,*

$$H_i(R_\bullet) = \begin{cases} H_i(N^{\#}_\bullet) & for \quad -(n-1) \leq i \leq n-1 \\ 0 & otherwise. \end{cases}$$

*which tells us that $R_\bullet$ has exactly two nonzero homologies less than than $P_\bullet$.*

**Proof.** Note that the only nonzero homology of $L_\bullet$ is at degree $n$. Using (5.8), the long exact homology sequence corresponding to the exact triangle is as follows:

$$\cdots H_{n+2}(R_\bullet) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_{n+1}(R_\bullet) \longrightarrow H_n(L_\bullet) \xrightarrow{\sim} H_n(N^{\#}_\bullet) \longrightarrow$$

$$H_n(R_\bullet) \longrightarrow 0 \longrightarrow H_{n-1}(N^{\#}_\bullet) \longrightarrow H_{n-1}(R_\bullet) \longrightarrow 0 \longrightarrow \cdots\cdots \longrightarrow$$

$$0 \longrightarrow H_{-n+1}(N^{\#}_\bullet) \longrightarrow H_{-n+1}(R_\bullet) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_{-n}(R_\bullet) \longrightarrow 0 \cdots$$

Therefore,

$$H_i(R_\bullet) = \begin{cases} H_i(N^{\#}_\bullet) & for \quad -(n-1) \leq i \leq n-1 \\ 0 & otherwise. \end{cases}$$

The proof is complete. $\square$

*Remark* 5.10. The readers are referred to [TWGI, 4.11] for the definition of a **very good morphism** $L_\bullet \longrightarrow N_\bullet^\#$. All we need in the sequel is that such morphisms exist [TWGI, 4.17] and that whenever they do, we obtain [TWGI, 4.20]

1. There is a symmetric form $\psi : R_\bullet \xrightarrow{\sim} R_\bullet^\#$.

2. There is a lagrangian

$$N_\bullet^\# \longrightarrow (P_\bullet^\#, \varphi^{-1}) \perp (R_\bullet, \psi).$$

**Theorem 5.11.** *Let $(P_\bullet, \varphi)$ be a symmetric form as in (5.3) with $n > 0$. Then, there is a symmetric form $(Q_\bullet, \tau)$ such that*

*1.*

$$[(Q_\bullet, \tau)] = [(P_\bullet, \varphi)] \qquad in \quad W\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{\pm}_u\right).$$

*2. $Q_\bullet$ has two less homologies than $P_\bullet$ and it has support in $[k+d, -k]$ for some $0 \le k < n$.*

**Proof.** Use the notations in (5.6),(5.7) and (5.8). Using the above remark (5.10), let $\mu_0$ be a very good morphism and let $(R_\bullet, \psi)$ be the symmetric form obtained. Note that $N_\bullet^\#, R_\bullet$ are objects in $D^b_{\mathcal{M}FPD^{fl}}(\mathcal{P}(A))$ by (5.7) and (5.9). Hence, by (2) of the above remark (5.10), $(R_\bullet, \psi)$ is Witt equivalent to $(P_\bullet^\#, -\varphi^{-1})$ in $D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{\pm}_u$ by definition (4.3). Therefore in $W\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{\pm}_u\right)$, we have

$$[(R_\bullet, \psi)] = [(P\bullet^\#, -\varphi^{-1})] = [(P_\bullet, -\varphi)] \quad \text{in} \quad W\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)^{\pm}_u\right).$$

Now, $H_i(R_\bullet) = 0$ if $i < -(n-1), i > (n-1)$. By (5.1), $(R_\bullet, -\psi)$ is isometric to a form $(Q_\bullet, \tau)$ such that $Q_\bullet$ is supported on $[k+d, -k]$ with $0 \le k \le n-1$. The proof is complete. □

Now we are ready to state and prove the main result of this article, which is our version of the dévissage theorem, i.e.

**Theorem 5.12.** *The homomorphisms*

$$W(\zeta) : W(\mathcal{A},^{\vee}, \pm\tilde{\varpi}) \longrightarrow W\left(D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u\right)$$

*induced by the functor* $\zeta$ *are isomorphisms.*

**Proof.** First, we prove the surjectivity of the homomorphism $W(\zeta)$. Suppose $x \in W\left(D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u\right)$. Then, by (5.1), we can write $x = [(P_\bullet, \varphi)]$ of the form described in (5.3). Inductively, by (5.11), there is a form $(R_\bullet, \psi)$ in $D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u$ such that

$$[(R_\bullet, \psi)] = [(P_\bullet, \varphi)] = x \qquad \textit{in} \quad W\left(D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u\right)$$

and $R_\bullet$ is supported in $[d, 0]$. By (2.9), $R_\bullet$ is a projective resolution of $M := H_0(R_\bullet) \in \mathcal{A}$. Further, $\psi$ induces a form $\psi_0 : M \stackrel{\sim}{\longrightarrow} M^{\vee}$ and clearly

$$W(\zeta)([(M, \psi_0)]) = [(R_\bullet, \psi)] = x.$$

So, $W(\zeta)$ is surjective.

Now, we proceed to prove that $W(\zeta)$ is injective. Suppose $(M, q)$ is a symmetric form in $(\mathcal{A},^{\vee}, \pm\tilde{\varpi})$ and $W(\zeta)([(M, q)]) = 0$. Write , $(\zeta(M), \zeta(q)) = (P_\bullet, \varphi_0)$ where $P_\bullet$ is a projective resolution of $M$ of length $d$, and $\varphi_0$ is the induced symmetric form. So, $[(P_\bullet, \varphi_0)] = 0$ in $W\left(D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u\right)$.

This means there is a neutral form $[(Q_\bullet, \varphi_1)]$ so that $[(P_\bullet, \varphi_0)] \perp [(Q_\bullet, \varphi_1)]$ is neutral in $W\left(D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u\right)$. Since $[(Q_\bullet, \varphi_1)]$ is neutral, so is $[(Q_\bullet, -\varphi_1)]$. Hence, $[(P_\bullet, \varphi_0)] \perp [(Q_\bullet, \varphi_1)] \perp [(Q_\bullet, -\varphi_1)]$ is neutral. Using the usual isometry, we get that there is a hyperbolic form

$$\left(Q_\bullet \oplus Q_\bullet^{\#}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \qquad \text{with} \quad Q_\bullet \in D^b_{\mathcal{A}}((\mathcal{P}(A))$$

such that

$$(U_\bullet, \varphi) := \left(P_\bullet \oplus Q_\bullet \oplus Q_\bullet^{\#}, \begin{pmatrix} \varphi_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \quad \text{is neutral in} \quad W\left(D^b_{\mathcal{A}}(\mathcal{A})^{\pm}_u\right)$$

(we have so far followed the argument in [TWGI, 3.5])
So, $(U_\bullet, \varphi)$ has a lagrangian $(L_\bullet, \alpha)$. We have an exact triangle

$$T^{-1}L_\bullet^{\#} \xrightarrow{w} L_\bullet \xrightarrow{\alpha} U_\bullet \xrightarrow{\alpha^{\#}\varphi} L_\bullet^{\#} \qquad T^{-1}w^{\#} = w.$$

Before proceeding, we use (3.3) to make a startlingly simple observation about the homologies of $L_\bullet^{\#}$ :

$$H_i(L_\bullet^{\#}) \cong H_{i-d}(L_\bullet^{*}) \cong Ext^d\left(H_{(d-i)-d}(L_\bullet), A\right) \cong Ext^d\left(H_{-i}(L_\bullet), A\right) \cong H_{-i}(L_\bullet)^\vee.$$

Similarly, $H_i(U_\bullet^{\#}) \cong H_{-i}(U_\bullet)^\vee$ and further using the remark (3.4) about naturality of the maps, we get that the long exact homology sequences are

$$\begin{array}{ccccccccc}
\longrightarrow & H_{-2}(L_\bullet)^\vee & \xrightarrow{H_1(w)} & H_1(L_\bullet) & \xrightarrow{H_1(\alpha)} & H_1(U_\bullet) & \xrightarrow{H_{-1}(\alpha)^\vee \circ H_1(\varphi)} & H_{-1}(L_\bullet)^\vee & \cdots\cdots \\
 & \| & & \tilde{\varpi}_{H_1} \downarrow \wr & & H_1(\varphi) \downarrow \wr & & \| & \\
\longrightarrow & H_{-2}(L_\bullet)^\vee & \xrightarrow[H_{-2}(w)^\vee]{} & H_1(L_\bullet)^{\vee\vee} & \xrightarrow[H_{-1}(\varphi)^\vee \circ H_1(\alpha)^{\vee\vee}]{} & H_{-1}(U_\bullet)^\vee & \xrightarrow[H_{-1}(\alpha)^\vee]{} & H_{-1}(L_\bullet)^\vee & \cdots\cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots\cdots & \xrightarrow{H_0(w)} & H_0(L_\bullet) & \xrightarrow{H_0(\alpha)} & H_0(U_\bullet) & \xrightarrow{H_0(\alpha)^\vee \circ H_0(\varphi)} & H_0(L_\bullet)^\vee & \xrightarrow{H_{-1}(w)} & \cdots\cdots \\
 & & \tilde{\varpi}_{H_0} \downarrow \wr & & H_0(\varphi) \downarrow \wr & & \| & & \\
\cdots\cdots & \xrightarrow[H_{-1}(w)^\vee]{} & H_0(L_\bullet)^{\vee\vee} & \xrightarrow[H_0(\varphi)^\vee \circ H_0(\alpha)^{\vee\vee}]{} & H_0(U_\bullet)^\vee & \xrightarrow[H_0(\alpha)^\vee]{} & H_0(L_\bullet)^\vee & \xrightarrow[H_0(w)^\vee]{} & \cdots\cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots\cdots H_{-1}(L_\bullet) & \xrightarrow{H_{-1}(\alpha)} & H_{-1}(U_\bullet) & \xrightarrow{H_1(\alpha)^\vee \circ H_{-1}(\varphi)} & H_{-1}(L_\bullet)^\vee & \xrightarrow{H_{-2}(w)} & H_{-2}(L_\bullet) & \xrightarrow{H_{-2}(\alpha)} & \\
\tilde{\varpi}_{H_{-1}} \downarrow \wr & & H_{-1}(\varphi) \downarrow \wr & & \| & & \tilde{\varpi}_{H_{-2}} \downarrow \wr & & \\
\cdots\cdots H_{-1}(L_\bullet)^{\vee\vee} & \xrightarrow[H_1(\varphi)^\vee \circ H_{-1}(\alpha)^{\vee\vee}]{} & H_1(U_\bullet)^\vee & \xrightarrow[H_1(\alpha)^\vee]{} & H_1(L_\bullet)^\vee & \xrightarrow[H_1(w)^\vee]{} & H_{-2}(L_\bullet)^{\vee\vee} & \xrightarrow[H_2(\varphi)^\vee \circ H_{-2}(\alpha)^{\vee\vee}]{} &
\end{array}$$

Replacing the part of the top exact sequence in negative degree by the corresponding part of the bottom (dual) exact sequence, we get an exact sequence :

$$\longrightarrow H_{-2}(L_\bullet)^\vee \xrightarrow{H_1(w)} H_1(L_\bullet) \xrightarrow{H_1(\alpha)} H_1(U_\bullet) \xrightarrow{H_{-1}(\alpha)^\vee \circ H_1(\varphi)} H_{-1}(L_\bullet)^\vee$$

$$\xrightarrow{H_0(w)} H_0(L_\bullet) \xrightarrow{H_0(\alpha)} H_0(U_\bullet) \xrightarrow[H_0(\alpha)^\vee \circ H_0(\varphi)]{} H_0(L_\bullet)^\vee \xrightarrow[H_0(w)^\vee]{} H_{-1}(L_\bullet)^{\vee\vee}$$

$$\xrightarrow[H_1(\varphi)^\vee \circ H_{-1}(\alpha)^{\vee\vee}]{} H_1(U_\bullet)^\vee \xrightarrow[H_1(\alpha)^\vee]{} H_1(L_\bullet)^\vee \xrightarrow[H_1(w)^\vee]{} H_{-2}(L_\bullet)^{\vee\vee} \xrightarrow[H_2(\varphi)^\vee \circ H_{-2}(\alpha)^{\vee\vee}]{}$$

Notice that the complex above is very special, and is "symmetric" about $H_0(U_\bullet)$. So we can apply [TWGII, 4.1 Lemma] to this sequence. Since the sequence is exact, we have

$$[(H_0(U_\bullet), H_0(\varphi))] = [(0,0)] = 0 \qquad in \quad W(\mathcal{M}FPD^{fl}(A)).$$

However,

$$(H_0(U_\bullet), H_0(\varphi)) = \left( H_0(P_\bullet) \oplus H_0(Q_\bullet) \oplus H_0(Q_\bullet^{\#}), \begin{pmatrix} H_0(\varphi_0) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right)$$

$$= (M, q) \perp \left( H_0(Q_\bullet) \oplus H_0(Q_\bullet)^\vee, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

So, we have

$$[(M,q)] = \left[ (M,q) \perp \left( H_0(Q_\bullet) \oplus H_0(Q_\bullet)^\vee, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right] = [(H_0(U_\bullet), H_0(\varphi))] = 0.$$

The proof is complete. □

# 6 Shifted Witt Groups

In this section, we use the previous results to obtain our dévissage theorem for the Witt groups $W^i(D^b_{\mathcal{A}}(\mathcal{P}(A)))$. We recall that $A$ is a Cohen-Macauly ring with $\dim A_m = d \geq 2$ for all maximal ideals $m$ and such that 2 is invertible in $A$ and that $\mathcal{A} = \mathcal{M}FPD^{fl}(A)$.

**Notations 6.1.** *For integers $j \geq 0$ define the functor $\zeta_j = T^{-j} \circ \zeta$, which associates to an object $M$ in $\mathcal{A}$ a projective resolution $P_\bullet$ of $M$ of length $d$, such that $H_{-j}(P_\bullet) = M$.*

**Definition 6.2.** Suppose $K := (K, \#, \delta, \varpi)$ is a triangulated category with translation $T$ and $\delta-$duality $\#$. We recall from [TWGII] that

$$T^n K := (K, T^n \circ \#, (-1)^n \delta), (-1)^{\frac{n(n+1)}{2}} \delta^n \varpi).$$

is then also a triangulated category with the same translation $T$ but with $((-1)^n\delta)$-duality $T^n \circ \#$. If $K_0$ is a subcategory of $K$ satisfying the conditions of (4.3), we define $T^n K_0$ to be the same subcategory and translation with the induced duality structure from $T^n K$. Using (4.3), we define the shifted Witt groups by

$$W^n(K) := W(T^n K) \qquad W^n(K_0) := W(T^n K_0) \qquad \forall\, n \in \mathbb{Z}.$$

Note that $T_s^2 : T^n K \longrightarrow T^{n+4} K$ is an equivalence of triangulated categories with duality, for all $n \in \mathbb{Z}$. Similarly, $T_s^2 : T^n K_0 \longrightarrow T^{n+4} K_0$ is an equivalence of categories with duality, for all $n \in \mathbb{Z}$ and so

$$W^n(K) \xrightarrow{\sim} W^{n+4}(K) \qquad W^n(K_0) \xrightarrow{\sim} (T^n K_0) \qquad \forall\, n \in \mathbb{Z}.$$

**Definition 6.3.** Following [BW], by **"standard" duality** structure on $\mathcal{A}$, we mean the exact category $\left(\mathcal{A},^{\vee}, (-1)^{\frac{d(d-1)}{2}} \tilde{\varpi}\right)$. By **"standard" skew duality** structure on $\mathcal{A}$, we mean the exact category $\left(\mathcal{A},^{\vee}, -(-1)^{\frac{d(d-1)}{2}} \tilde{\varpi}\right)$. We denote the Witt groups

$$W^+_{St}(A) = W\left(\mathcal{A},^{\vee}, (-1)^{\frac{d(d-1)}{2}} \tilde{\varpi}\right), \quad W^-_{St}(A) = W\left(\mathcal{A},^{\vee}, -(-1)^{\frac{d(d-1)}{2}} \tilde{\varpi}\right).$$

**Theorem 6.4.** *Then, the functor $\zeta_0 : \mathcal{A} \longrightarrow D^b_{\mathcal{A}}(\mathcal{P}(A))$ induces an isomorphism*

$$W(\zeta_0) : W^+_{St}(A) \xrightarrow{\sim} W^d\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), *, 1, \varpi\right).$$

**Proof.** Recall $\zeta_0$ was denoted by $\zeta$ in the previous sections. For notational convenience $\varpi_0 = (-1)^{\frac{(d(d-1)}{2}} \varpi$. By theorem (5.12), we get the following isomorphism of Witt groups

$$\eta_0 : W^+_{St}(\mathcal{A}) \xrightarrow{\sim} W\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), \#^u_d, 1, \varpi_0\right).$$

There is a duality preserving equivalence [BW, Proof of Lemma 6.4]

$$\beta : \left(D^b(\mathcal{P}(A)), T^d_{u^\circ} *, 1, \varpi_0\right) \longrightarrow \left(D^b(\mathcal{P}(A)), T^d_{u^\circ} *, (-1)^d, (-1)^{\frac{d(d+1)}{2}} \varpi\right).$$

Note that the later is the shifted category $T^d\left(D^b(\mathcal{P}(A)), *, 1, \varpi\right)$. Composing $\eta_0$ with the homomorphism induced by $\beta$, we get the result. $\square$

Now we prove the standard skew duality version of theorem (6.4).

**Theorem 6.5.** *The functor $\zeta_1$ induces an isomorphism*

$$W^-_{St}(A) \xrightarrow{\sim} W^{d-2}\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), *, 1, -\varpi\right).$$

**Proof.** By Theorem (5.12), we have an isomophism

$$W^-_{St}(A) \xrightarrow{\sim} W\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T^d_u \circ *, 1, -(-1)^{\frac{d(d-1)}{2}}\varpi\right).$$

Write $\varpi_0 = -(-1)^{\frac{d(d-1)}{2}}\varpi$. There is an equivalence of categories [TWGI, 2.14]

$$T_s : \left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T^{d-2}_u \circ *, 1, \varpi_0\right) \longrightarrow \left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T^d_u \circ *, 1, \varpi_0\right).$$

This induces an isomorphism

$$W\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T^{d-2}_u \circ *, 1, \varpi_0\right) \xrightarrow{\sim} W\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T^d_u \circ *, 1, \varpi_0\right).$$

As in the proof of [BW, Lemma 6.4], there is an equivalence of triangulated categories with duality

$$\left(D^b(\mathcal{P}(A)), T^{d-2}_u \circ *, 1, \varpi_0\right) \longrightarrow \left(D^b(\mathcal{P}(A)), T^{d-2}_s \circ *, (-1)^{d-2}, (-1)^{\frac{d(d+1)}{2}}\varpi\right).$$

The latter category is $T^{d-2}\left(D^b(\mathcal{P}(A)), *, 1, -\varpi\right)$. The proof is complete. □

Finally, we have the following regarding odd shifts.

**Theorem 6.6.** *For $n = d-1, d-3$, we have*

$$W^n\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), *, 1, \pm\varpi\right) = 0.$$

**Proof.** First consider $n = d-1$. It would be enough to prove that

$$W\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T^{d-1}_u \circ *, 1, \pm\varpi\right) = 0.$$

Suppose $(P_\bullet, \varphi)$ is a form in $D^b_{\mathcal{A}}\left(\mathcal{P}(A), T^{d-1}_u \circ *, 1, \pm\varpi\right)$. By a little tweak in (5.1), we can assume that $P_\bullet$ is supported on $[n + (d-1), -n]$ with $n > 0$ and $H_{-n}(P_\bullet) \neq 0$. By imitating the arguments of theorem (5.11), we can keep shortening the length of the complexes which give our symmetric form. Eventually, we will be reduced to the case where the complex is $P_\bullet$ is supported on $[d-1, 0]$. By theorem (2.9), $P_\bullet$ is exact. So, $[(P, \varphi)] = 0$. The same arguments apply when $n = d-3$. The proof is complete. □

Using the 4-periodicity, we now obtain the theorem mentioned in the introduction :

**Theorem 6.7** (shiftFinal)**.** *Let* $\mathcal{B} = \left(D^b_{\mathcal{A}}(\mathcal{P}(A)), T_s,^*, 1, \varpi\right)$. *Then, for* $n \in \mathbb{Z}$*, we have*

1. $W^{d+4n}(\mathcal{B}) = W_s^+(\mathcal{A})$,
2. $W^{d+4n+1}(\mathcal{B}) = 0$,
3. $W^{d+4n+2}(\mathcal{B}) = W_s^-(\mathcal{A})$,
4. $W^{d+4n+3}(\mathcal{B}) = 0$.

# A Some Formalism

The purpose of this section is to prove the following theorem :

**Theorem A.1.** *Suppose* $\mathcal{E}$ *is a full subcategory of a* $\mathbb{Z}[\frac{1}{2}]$ *abelian category* $\mathcal{B}$ *with the* 2 *out of* 3 *property for short exact sequences, and has duality* $(\mathcal{E},^\vee, \tilde{\varpi})$. *Let* $D^b(\mathcal{E}) := \left(D^b(\mathcal{E}), *, a, \varpi\right)$ *denote the derived category, with duality, of* $(\mathcal{E},^\vee, \tilde{\varpi})$. *Also, let* $D^b_{\mathcal{E}}(\mathcal{E})$ *denote the derived category, with duality, of objects in* $D^b(\mathcal{E})$ *with homologies in* $\mathcal{E}$. *Then the homomorphism*

$$W(\mu) : W(\mathcal{E},^\vee, \tilde{\varpi}) \longrightarrow W\left(D^b_{\mathcal{E}}(\mathcal{E})\right)$$

*induced by the functor* $\mu : \mathcal{E} \longrightarrow D^b_{\mathcal{E}}(\mathcal{E})$ *is an isomorphism.*

In particular, with $\mathcal{E} = \mathcal{A}$ and $\mathcal{B} = \mathcal{M}(A)$, we obtain that

$$W(\mu) : W(\mathcal{A},^\vee, \pm\tilde{\varpi}) \longrightarrow W\left(D^b_{\mathcal{A}}(\mathcal{A},^\vee, \pm\tilde{\varpi})\right)$$

as promised in section 5.

The proof of the theorem is essentially the same as the proof of [TWGII, Theorem 3.2] with the extra check that all constructions yield complexes whose homologies are in $\mathcal{A}$. This boils down to using the most elementary of sublagrangians (concentrated in just one degree) and reducing length. In any case, we follow the proof in [TWGII, Theorem 3.2]. Since the category

$\mathcal{E}$ has all the properties required in the results in [TWGII, Section 3]), we will freely borrow them.

To start with, injectivity of $W(\mu)$ follows directly because the isomorphism $W(\mathcal{E}) \xrightarrow{\sim} W\left(D^b(\mathcal{E})\right)$ (proven in [TWGII, Theorem 4.3]) factors as

$$\begin{array}{ccc} W(\mathcal{E}) & \xrightarrow{W(\mu)} & W\left(D^b_{\mathcal{E}}(\mathcal{E})\right) \\ & \searrow^{\sim} & \downarrow \\ & & W\left(D^b(\mathcal{E})\right) \end{array}$$

We move to the proof of surjectivity which, as we mentioned above will require checking that we remain in $D^b_{\mathcal{E}}(\mathcal{E})$ through all the lemmas establishing [TWGII, Theorem 3.2]. To start with, we establish the following result regarding duality, which also provides an alternative proof of (3.5).

**Lemma A.2.** *With the same notations as in (A.1), $D^b_{\mathcal{E}}(\mathcal{E})$ is closed under duality.*

**Proof.** Let $P_\bullet$ be an object in the derived category $D^b_{\mathcal{E}}(\mathcal{E})$. Write

$$P_\bullet : \qquad \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

$$P^*_\bullet : \qquad \cdots \longrightarrow P^\vee_{-2} \longrightarrow P^\vee_{-1} \longrightarrow P^\vee_0 \longrightarrow P^\vee_1 \longrightarrow P^\vee_2 \longrightarrow \cdots$$

Since the complexes are bounded and the homologies are objects of $\mathcal{E}$, all the kernels $Z_i = ker(d_i)$, images $B_i = image(d_{i+1})$ and quotients $\frac{P_i}{B_i}$ are also objects of $\mathcal{E}$. Hence, so are their duals. But we have an exact sequence

$$0 \longrightarrow \left(\frac{P_{t-1}}{B_{t-1}}\right)^\vee \longrightarrow P^\vee_{t-1} \longrightarrow \left(\frac{P_t}{B_t}\right)^\vee \longrightarrow H_t(P^*_\bullet) \longrightarrow 0$$

The first three terms in this sequence are in $\mathcal{E}$, hence so is $H_0(P^*_\bullet)$. The proof is complete. $\square$

**Lemma A.3.** *Let $x \in W\left(D^b_{\mathcal{E}}(\mathcal{E})\right)$. Then $x = (P_\bullet, s)$ such that*

1. *$P_\bullet$ is bounded and $s : P_\bullet \longrightarrow P^*_\bullet$ is a morphism of complexes, without denominator.*

2. *$s$ is quasi-isomorphism.*

3. *$s$ is strongly symmetric (i.e $s^{\vee}_{-i} = s_i$ $\forall i \in \mathbb{Z}$).*

4. *$H_i(P_\bullet) \in \mathcal{E}$ for all $i \in \mathbb{Z}$.*

**Proof.** Let the form $x$ be given by $(X_\bullet, \eta)$. By definition there is a complex $P_\bullet$ which is an object of $D^b_{\mathcal{E}}(\mathcal{E})$ and a chain complex quasi-isomorphisms $t : P_\bullet \longrightarrow X_\bullet$ and $\varphi_0 : P_\bullet \longrightarrow X^*_\bullet$ such that $\eta = \varphi_0 t^{-1}$. Then, $s = t^*\varphi_0 = t^*\eta t$ is a symmetric form on $P_\bullet$ and $(X_\bullet, \eta)$ is isometric to $(P_\bullet, \varphi)$. Clearly $s$ is an actual morphism of complexes, a quasi-isomorphism and $H_i(P_\bullet) \in \mathcal{E}$ for all $i \in \mathbb{Z}$. Finally, using that $\frac{1}{2}$ exists, we can make the map strongly symmetric. □

**Lemma A.4.** *Let $(P_\bullet, s)$ be a symmetric form in $D^b_{\mathcal{E}}(\mathcal{E})$ as in (A.3), such that $P_\bullet$ is supported on $[m, -n]$ with $m > n \geq 0$. Then $(P_\bullet, s)$ is isometric to a symmetric space $(Q_\bullet, t)$ such that $Q_\bullet$ is supported on $[n, -n]$ and $(Q_\bullet, t)$ has all the other properties of $(P_\bullet, s)$.*

**Proof.** This is precisely [TWGII, Lemma 3.7] in our context. Note that since we can do this in the derived category without the homology condition $D^b(\mathcal{E})$, we use the same result to get $(Q_\bullet, t)$ isometric to $(P_\bullet, s)$ with the required condition. However, since the isometry gives in particular a quasi-isomorphism $P_\bullet \xrightarrow{\sim} Q_\bullet$ and so $Q_\bullet$ is also an object in $D^b_{\mathcal{E}}(\mathcal{E})$. The proof is complete. □

**Lemma A.5.** *Let $(P_\bullet, s)$ be a symmetric space, as in (A.3). with support on $[-n, n]$ and $n > 0$. Then there exists a symmetric space $(Q_\bullet, t)$ such that*

1. *$(Q_\bullet, t)$ is as in (A.3).*

2. *$(Q_\bullet, t)$ is supported in $[n, -(n-1)]$.*

3. *$H_i(Q_\bullet) \in \mathcal{E}$ for all $i \in \mathbb{Z}$.*

4. *$[(P_\bullet, s)] + [(Q_\bullet, t)] = 0$ in $W\left(D^b_{\mathcal{E}}(\mathcal{E})\right)$.*

**Proof.** Once again this is [TWGII, Lemma 3.9] in our context. We skim through the proof mentioning only the significant points and most important, checking the points where we need to check the extra homology condition. We begin by proving the lemma in the case $n \geq 2$. Write

$$
\begin{array}{ccccccccccccc}
P_\bullet = & \cdots & 0 & \longrightarrow & P_n & \xrightarrow{d} & P_{n-1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & P_{-n} & \longrightarrow & 0 \\
\downarrow s & & & & \downarrow s & & \downarrow s & & & & \downarrow s & & \\
P_\bullet^* = & \cdots & 0 & \longrightarrow & P_{-n}^\vee & \xrightarrow[d^\vee]{} & P_{-(n-1)}^\vee & \xrightarrow[d^\vee]{} & \cdots & \longrightarrow & P_n^\vee & \longrightarrow & 0
\end{array}
$$

Define $(Q_\bullet, t)$ as follows, on the left side:

$$
\begin{array}{ccccccccccccccc}
Q_\bullet = \cdots & 0 & \longrightarrow & P_n & \xrightarrow{\begin{pmatrix} s \\ d \end{pmatrix}} & P_{-n}^\vee \oplus P_{n-1} & \xrightarrow{(0,d)} & P_{n-2} & \xrightarrow{d} \cdots & P_{-(n-2)} & \xrightarrow{d} & P_{-(n-1)} & \longrightarrow & 0 & \longrightarrow 0 \\
\downarrow t & & & \downarrow 0 & & \downarrow (d^\vee, -s) & & \downarrow -s & & \downarrow -s & & \downarrow \begin{pmatrix} d \\ -s \end{pmatrix} & & & \\
Q_\bullet^* = \cdots & 0 & \longrightarrow & 0 & \xrightarrow[d^\vee]{} & P_{-(n-1)}^\vee & \xrightarrow[d^\vee]{} & P_{-(n-2)}^\vee & \xrightarrow[d^\vee]{} \cdots & P_{(n-2)}^\vee & \xrightarrow[\begin{pmatrix} 0 \\ d^\vee \end{pmatrix}]{} & P_{-n} \oplus P_{n-1}^\vee & \xrightarrow[(s,d^\vee)]{} & P_n^\vee & \longrightarrow 0
\end{array}
$$

It was proved in [TWGII, Lemma 3.9] that $t$ is a quasi-isomorphism. So, it follows

$$H_n(Q_\bullet) \cong 0, \quad H_{n-1}(Q_\bullet) \cong H_{n-1}(Q_\bullet^*) \cong \ker(d^\vee) \in \mathcal{E}.$$

Since $\text{image}(0, d_{n-1}) = \text{image}(d_{n-1})$ we have

$$H_i(Q_\bullet) = H_i(P_\bullet) \in \mathcal{E} \qquad \forall \quad i \leq n-2.$$

Therefore

$$H_i(Q_\bullet) \in \mathcal{E} \qquad for\ all \quad i \in \mathbb{Z}.$$

So $Q_\bullet$ satisfies the last condition of (A.3). The other conditions of (A.3) are shown to be established in [TWGII, Lemma 3.9].

Therefore, $Q_\bullet$ satisfies (A.3). It was established in [TWGII] that $(P_\bullet, s) \perp (Q_\bullet, t)$ is neutral in $D^b(\mathcal{E})$, by showing that $(P_\bullet, s) \perp (Q_\bullet, t)$ is isometric (in

$D^b(\mathcal{E})$) to the cone of the morphism $z : T^{-1}M_\bullet^* \longrightarrow M_\bullet$ defined as follows:

$$\begin{array}{ccccccccccc}
T^{-1}M_\bullet^* = \cdots 0 & \longrightarrow & P_n & \xrightarrow{-d} & P_{n-1} & \xrightarrow{-d} \cdots & P_{-(n-2)} & \xrightarrow{-d} & P_{-(n-1)} & \longrightarrow & 0 \\
\downarrow z & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow s_{-n}d & & \\
M_\bullet = \cdots 0 & \longrightarrow & P^\vee_{-(n-1)} & \xrightarrow[d^\vee]{} & P^\vee_{-(n-2)} & \xrightarrow[d^\vee]{} \cdots & P^\vee_{n-1} & \longrightarrow & P^\vee_n & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & & & \uparrow \\
degree = & & n & & n-1 & & & & & & -n
\end{array}$$

Since all the boundaries and cycles of $P_\bullet$ and $P_\bullet^*$ are objects of $\mathcal{E}$, so are $H_i(M_\bullet)$ and $H_i(M_\bullet)^*$. Therefore,$M_\bullet$ and $M_\bullet^*$ are objects of $D^b_{\mathcal{E}}(\mathcal{E})$.

Let $Z_\bullet = cone(z)$. In [TWGII, Lemma 3.9], it is further shown that there is a symmetric form $\chi : Z_\bullet \longrightarrow Z_\bullet^*$ and that $(Z_\bullet, \chi)$ is isometric to $(P_\bullet, s) \perp (Q_\bullet, t)$ in $D^b(\mathcal{E})$. But this tells us that $H_i(Z_\bullet) \cong H_i(P_\bullet) \oplus H_i(Q_\bullet)$ and hence $Z_\bullet$ is an object of $D^b_{\mathcal{E}}(\mathcal{E})$.

Now again following the proof of [TWGII, Lemma 3.9], it is shown that $T^{-1}z^\# = z$ in $D^b_{\mathcal{E}}(\mathcal{E})$ and that the form $\chi$ actually fits in to make $M_\bullet$ a lagrangian for $(Z_\bullet, \chi)$. Hence, this proves the lemma when $n \geq 2$.

In the case $n = 1$, $(P_\bullet, s)$ is given by

$$\begin{array}{ccccccccc}
P_\bullet = & 0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & P_{-1} & \longrightarrow 0 \\
\downarrow s & & & \downarrow s & & \downarrow s & & \downarrow s & \\
P_\bullet^* = & 0 & \longrightarrow & P^\vee_{-1} & \longrightarrow & P^\vee_0 & \longrightarrow & P^\vee_1 & \longrightarrow 0
\end{array}$$

Define $(Q_\bullet, s)$ as follows

$$\begin{array}{ccccccccc}
Q_\bullet = \cdots 0 & \longrightarrow & P_1 & \xrightarrow{\begin{pmatrix} s \\ d \end{pmatrix}} & P^\vee_{-1} \oplus P_0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow t & & \downarrow & & \downarrow \begin{pmatrix} 0 & d \\ d^\vee & -s \end{pmatrix} & & \downarrow & & \\
Q_\bullet^* = \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_{-1} \oplus P^\vee_0 & \xrightarrow[\begin{pmatrix} s^\vee d^\vee \end{pmatrix}]{} & P^\vee_1 & \longrightarrow & 0
\end{array}$$

The degree zero term is in the middle. In [TWGII, Lemma 3.9], it is established that $t$ is a quasi-isomorphism. It follows that $H_i(Q_\bullet) = 0$ for all $i \neq 0$ and

$$H_0(Q_\bullet) = \frac{P_{-1}^\vee \oplus P_0}{P_1} \in \mathcal{E}.$$

So, $Q_\bullet$ satisfies all the condition in (A.3), because the remaining three conditions are established in [TWGII, Lemma 3.9]. Now $(P, s) \perp (Q, t)$ has a lagrangian, namely

$$\begin{array}{ccccccccc}
T^{-1}M_\bullet^* = & & 0 & \longrightarrow & P_1 & \xrightarrow{-d} & P_0 & \longrightarrow & 0 \\
{\scriptstyle z}\downarrow & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle sd} & & \\
M_\bullet = & & 0 & \longrightarrow & P_0^\vee & \xrightarrow[d^\vee]{} & P_1^\vee & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
degree = & & 1 & & 0 & & -1 & &
\end{array}$$

Again,

$$H_0(M_\bullet) = \ker(d^\vee), \quad H_1(M_\bullet) = \operatorname{coker}(d^\vee) \quad are\ in \quad \mathcal{E}.$$

So, $M_\bullet$ and hence $M_{\bullet^*}$ are objects of $D^b_{\mathcal{E}}(\mathcal{E})$. The rest of the proof is exactly the same as in the case $n \geq 2$. The proof is complete. □

**Finishing the prof of (A.1)** :
We use (A.3) to represent any element $x$ in $W(D^b_{\mathcal{A}}(\mathcal{A}))$ by a chain complex in $D^b_{\mathcal{A}}(\mathcal{A})$ and a strongly symmetric quasi-isomorphism to its dual. Then, by alternate use of lemma (A.4) and (A.5), we reduce any element in $W\left(D^b_{\mathcal{E}}(\mathcal{E})\right)$ to a chain complex in $D^b_{\mathcal{E}}(\mathcal{E})$), concentrated at degree zero. Of course that means the quasi-iromorphism is actually an isomorphism and hence $x$ is the image of an element in $W(\mathcal{A})$ via $W(\mu)$. So $W(\mu)$ is also surjective. So, the proof of theorem (A.1) is complete. □

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