REDUCERS AND K_0 WITH SUPPORT

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Abstract. Let R be a commutative noetherian ring, $\mathscr{L}$ be a Serre subcategory of the category of finitely generated R-modules and $\mathcal{P}$ be the category of finitely generated projective R-modules. We define invariants based on the categories of chain complexes from $\mathcal{P}$ homologically supported in $\mathscr{L}$ with support between 0 and n ($\{Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\}_{n\in\mathbb{N}}$) and the K-group with support $K_0(R \text{ on } \mathscr{L})$ similar to the stable range in classical K-theory. We introduce a notion called a reducer which we use to express the class of a complex in terms of classes of complexes of smaller amplitude and use these to study and bound the above defined invariants by standard invariants like arithmetic rank, grade and projective dimension. We also give conditions for $K_0(R \text{ on } \mathscr{L})$ to be isomorphic to K_0(modules in $\mathscr{L}$ with finite projective dimension).

1. Introduction

Let R be a commutative noetherian ring, $\mathcal{M}$ be the category of finitely generated R-modules and $\mathcal{P}$ be the full subcategory of projective R-modules. In classical K-theory, the group $\tilde{K_0}(R)$ can be obtained as the direct limit of the pointed sets $\Upsilon_k = \{P \in \mathcal{P} | rk(P) = k\}$ via the maps from $\Upsilon_k \to \Upsilon_{k+1}$ sending P to $P \oplus R$. Classical problems of splitting a free summand and cancellation can then be viewed as asking for the surjectivity and injectivity, respectively, of these maps. Similarly, $K_1(R)$ is the direct limit of $GL_n(R)/E_n(R)$ and the stable range in this case has been studied in great detail.

In this article, we study a "derived" version of these stable ranges. Let $\mathscr{L}$ be a Serre subcategory of $\mathcal{M}$ and define $K_0(R \text{ on } \mathscr{L})$ as $K_0(D^b_{\mathscr{L}}(\mathcal{P}))$. Consider the exact category of chain complexes $Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})$ (which is closed under kernels of surjections). We show in (2.15) that

Lemma.

$$\varinjlim \frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact} \rangle} \xrightarrow{\sim} \frac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact} \rangle} \xrightarrow{\sim} \frac{K_0\left(Ch^{b}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact} \rangle} \xrightarrow{\sim} K_0(R \text{ on } \mathscr{L}).$$

This description of $K_0(R \text{ on } \mathscr{L})$ along with the previously mentioned classical notions motivates the natural question(s) which are central to this article :

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Question 1. When is the natural map $\frac{K_0\left(Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \xrightarrow{\mathcal{I}'_k} \frac{K_0\left(Ch^{[0,k+1]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$ surjective (resp. injective)?

The previous lemma and the above question are implicitly observed in [7], wherein the map $\mathcal{I}'_n$ is proved to be an isomorphism for certain $\mathscr{L}$ and n large enough. Thus, we define $\alpha_s(\mathscr{L})$ $(/\alpha_i(\mathscr{L})/\alpha(\mathscr{L}))$ to be the least number n for which $\mathcal{I}'_k$ is surjective (resp. injective/isomorphic) for all $k \geq n$. Clearly, $\alpha = \max\{\alpha_s, \alpha_i\}$. By the well known bijection between specialization closed subsets of $Spec(R)$ and Serre subcategories of $\mathcal{M}$, these give in particular an interesting invariant for closed sets. In this article, we explore the nature of these invariants by means of a notion which we call reducers, to answer question 1. For a given complex $P.$, a reducer is a complex of smaller amplitude using which one can write the class $[P.]$ in terms of classes of complexes of smaller amplitude (3.1). In theorem 3.5 , we prove :

Theorem. *Let $\mathscr{L} \subset \mathcal{M}$ be a Serre subcategory. If every $P. \in Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ has a reducer, then $\mathcal{I}'_n$ is an isomorphism.*

We define $\beta(\mathscr{L})$ to be the least n such that every $P. \in Ch^{[0,m]}_{\mathscr{L}}(\mathcal{P}), m > n$ has a reducer. Clearly $\alpha(\mathscr{L}) \leq \beta(\mathscr{L})$. The primary example of a reducer is the Koszul construction at the heart of the proof of the main theorem in [7]. We use it to show that $\beta(\mathscr{L})$ is further bounded by the arithmetic rank (as an interesting aside, we generalize arithmetic rank to Serre subcategories). We make use of a different set of reducers when $\mathscr{L} = \mathscr{L}_{V(I)}$ where I is a monomial ideal or $char(R) = p$ and I has finite projective dimension, to show that $\beta(\mathscr{L})$ is bounded by the projective dimension of I. This links the invariants we have defined to standard invariants from commutative algebra. We also compute these invariants for some Serre subcategories.

We pause in our description of this article to briefly sketch the historical context of question 1. Let $\mathcal{H}$ be the full subcategory of $\mathcal{M}$ consisting of modules with finite projective dimension. Let $S \subset R$ be a multiplicatively closed set. Classical K-theory yields the fundamental exact sequence $K_1(R) \to K_1(S^{-1}R) \to K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R)$ where $K_0(R \text{ on } S)$ is "K-theory with support" as described in [1]. Let $\mathscr{L}_S$ be the Serre subcategory of $\mathcal{M}$ consisting of S-torsion modules (i.e. modules M s.t. $sM = 0$ for some $s \in S$), $\mathcal{H}_S = \mathcal{H} \cap \mathscr{L}_S$ and $\mathcal{H}_{n,S}$ be the full subcategory of $\mathcal{H}_S$ consisting of modules with projective dimension at most n. If S consists of non zero divisors, $K_0(R \text{ on } S)$ is isomorphic to $K_0(\mathcal{H}_S)$ and the proof [3, Theorem 4.4] shows an isomorphism with $K_0(\mathcal{H}_{1,S})$. Reinterpreting the result yields that $\alpha(\mathscr{L}_S) \leq 1$ in general and when S consists of non-zero divisors, $\alpha_s(\mathscr{L}_S) = \alpha(\mathscr{L}_S) = 1$. Note that $\alpha_i(\mathscr{L}_S) = 0$ always holds.

Subsequent to the formal definitions of algebraic K-theory and its properties in [15], [23] and [21], the above sequence can be identified as the end of the localization exact sequence $\ldots \to K_{i+1}(S^{-1}R) \to K_i(R \text{ on } S) \to K_i(R) \to K_i(S^{-1}R) \ldots$ [24, V Thm. 2.6.3]. Let $D^b(\mathcal{P})$ be the bounded derived category of $\mathcal{P}$ and $D^b_{\mathscr{L}_S}(\mathcal{P})$ its subcategory consisting of complexes $P.$ such that $S^{-1}P.$ is exact. $K_i(R \text{ on } S)$ is defined as $K_i(D^b_{\mathscr{L}_S}(\mathcal{P}))$ where $D^b_{\mathscr{L}_S}(\mathcal{P})$ and all other derived categories appearing in this article are thought of as localizations obtained from the usual model structure on the corresponding chain complex category (i.e. as a Waldhausen category). Then $K_0(R \text{ on } S) := K_0(D^b_{\mathscr{L}_S}(\mathcal{P}))$ is generated by complexes concentrated in degrees 0 and 1 [24, III Lemma 3.1.5] connecting it with the earlier definition and showing that $\alpha_s(\mathscr{L}_S) \leq 1$. Further, when S consists of non zero divisors, $K_i(R \text{ on } S) \simeq K_i(\mathcal{H}_S) \simeq K_i(\mathcal{H}_{1,S}) \forall i$ [24, II Ex. 9.13] where the last isomorphism follows from the resolution theorem [24, II Corollary 7.7.3] recovering the earlier result.

Let $X = Spec(R), U \subseteq X$ be an open subset and $Z = V(I)$ be its complement. Let $\mathscr{L}_Z$ denote the Serre subcategory of $\mathcal{M}$ consisting of modules supported on Z, $\mathcal{H}_Z = \mathcal{H} \cap \mathscr{L}_Z$ and $\mathcal{H}_{n,Z}$ be the full subcategory of $\mathcal{H}_Z$ consisting of modules with projective dimension at most n. The localization theorem yields a long exact sequence $\ldots \to K_{i+1}(U) \to K_i(X \text{ on } Z) \to K_i(X) \to K_i(U) \ldots$, where the third term is defined as $K_i(X \text{ on } Z) := K_i(D^b_Z(\mathcal{P})) = K_i(D^b_{\mathscr{L}_Z}(\mathcal{P}))$. When R is regular, by classical results, this term is isomorphic, via an equivalence of the underlying derived categories (or rather their model structures), to $G_i(Z) := K_i(\mathscr{L}_Z)$. When I is a local complete intersection ideal, this term can be identified with $K_i(\mathcal{H}_Z)$ [21].

When $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d, it was shown in [17, Prop 2] that $K_0(R \text{ on } \{\mathfrak{m}\}) \simeq K_0(\mathcal{H}_{\{\mathfrak{m}\}})$. The proof relies on the Koszul complex construction (an initial version of which appears in the preprint [6]) mentioned earlier. The result was subsequently generalized in [7] by considering several multiplicatively closed sets $S_1, S_2, \ldots S_d$ and the Serre subcategory $\mathscr{L} = \cap \mathscr{L}_{S_i}$ to combine the descriptions from the two seemingly separate cases of $K_0(R \text{ on } S)$ and $K_0(R \text{ on } Z)$. In essence, it is shown that $\beta(\mathscr{L}) \leq d$. In special cases when R is local, this is used to conclude that $K_0(R \text{ on } Z) \simeq K_0(\mathcal{H}_{n,Z})$ for some n and hence in particular that $K_0(R \text{ on } Z) \simeq K_0(\mathcal{H}_Z)$ explaining the Cohen-Macaulay result. Indeed, this article is motivated by [7] (which in turn is motivated by [17, Prop 2]).

The above historical context suggests the following well known question which has been studied classically since derived and triangulated categories first appeared.

Question 2. What conditions on $\mathscr{L}$ ensure that $K_0(R \text{ on } \mathscr{L})$ is isomorphic to $K_0(\mathcal{H} \cap \mathscr{L})$?

Question 2 is related to the new intersection theorem, which determines the smallest amplitude of a complex with non-zero finite length homology over a local ring. We answer question 2 in theorem 6.3 :

Theorem. *If $grade(\mathscr{L}) = \beta(\mathscr{L})$ then $K_0(R \text{ on } \mathscr{L}) \simeq K_0(\mathcal{H} \cap \mathscr{L})$.*

As a consequence, we obtain most of the above described results for K_0 in the affine case, and further that when $\mathscr{L} = \mathscr{L}_{V(I)}$ where I is a set theoretic complete intersection, $K_0(R \text{ on } \mathscr{L}) \simeq K_0(\mathcal{H} \cap \mathscr{L})$ recovering a very special case of more general results in [20].

We describe below the layout of the article. In section 2 we collect together all the basic definitions and results required for the article. In particular, we define the invariants $\alpha, \alpha_i, \alpha_s$, codimension and grade for Serre subcategories. In section 3, we introduce and study reducers, and define $\beta(\mathscr{L})$. In section 4, we define the arithmetic rank, place and improve the results in [7] by viewing the Koszul construction in [7] as a reducer and obtain the arithmetic rank as a bound for $\beta(\mathscr{L})$. We also compute all the invariants for some Serre subcategories and observe that condition (*) from [20] can be re-interpreted as $grade(\mathscr{L}) = ara(\mathscr{L})$ (which can thus be thought of as set-theoretic complete intersection for Serre subcategories). In subsection 5.1, when $\mathscr{L} = \mathscr{L}_{V(I)}$, we formulate suitable hypotheses (specifically that there are reducers with suitable properties) to deduce a refined bound for $\beta(\mathscr{L})$ in terms of projective dimension and in subsection 5.2 , we show that these hypotheses are satisfied when either I is a monomial ideal or when $char(R) = p$ and I has finite projective dimension. Monomial ideals then provide us with examples where $\alpha(\mathscr{L}_Z) < ara(Z)$. Finally, in section 6 , we briefly discuss α_i and then answer question 2.

2. Preliminaries

Throughout the article, R is a commutative, noetherian ring and all modules that occur are finitely generated R-modules.

2.1. **Preliminaries on categories.** We fix the following notations :
$\mathcal{M}$: the category of finitely generated R-modules
$\mathcal{P}$: the full subcategory of projective R-modules in $\mathcal{M}$.
$\mathcal{H}$: the full subcategory of R-modules with finite projective dimension in $\mathcal{M}$.
$Ch^b(\mathcal{P})$: the category of bounded chain complexes of R-modules in $\mathcal{P}$.
For $P. \in Ch^b(\mathcal{P})$, let
$\max(P.) = \max\{m \in \mathbb{Z} \mid P_m \neq 0\}$ (note $\max(0.) = -\infty$)
$\min(P.) = \min\{m \in \mathbb{Z} \mid P_m \neq 0\}$ (note $\min(0.) = \infty$)
$amplitude(P.) = \max(P.) - \min(P.)$ (note $amplitude(0.) = -\infty$).
$Ch^{[0,l]}(\mathcal{P})$: the full subcategory of $Ch^b(\mathcal{P})$ of complexes $P.$ such that $\min(P.) \geq 0$ and $\max(P.) \leq l$.
$Ch^{[0,\infty)}(\mathcal{P}) = \cup Ch^{[0,l]}(\mathcal{P})$.
$Ch^{h[0,l]}(\mathcal{P})$: the full subcategory of $Ch^b(\mathcal{P})$ of complexes $P.$ with the condition that for $i < 0$ and $i > l$, $H_i(P.) = 0$.
$D^b(\mathcal{P})$: the bounded derived category of $\mathcal{P}$, obtained by inverting quasi-isomorphisms in $Ch^b(\mathcal{P})$. This is a triangulated category.

Definition 2.1. A Serre subcategory $\mathscr{L}$ of an abelian category is a full subcategory such that for a short exact sequence $0 \to M'' \to M \to M' \to 0$ in the ambient category, $M \in \mathscr{L}$ iff $M', M'' \in \mathscr{L}$.

For a Serre subcategory $\mathscr{L}$ of $\mathcal{M}$, let $Ch^b_{\mathscr{L}}(\mathcal{P})$ (resp. $Ch^{[0,l]}_{\mathscr{L}}(\mathcal{P}), Ch^{h[0,l]}_{\mathscr{L}}(\mathcal{P})$) be the full subcategory of $Ch^b(\mathcal{P})$ (resp. $Ch^{[0,l]}(\mathcal{P}), Ch^{h[0,l]}(\mathcal{P})$) with homologies in $\mathscr{L}$ and $D^b_{\mathscr{L}}(\mathcal{P})$ be its derived category, i.e. the triangulated category obtained by inverting quasi-isomorphisms. Note that this is a thick subcategory of $D^b(\mathcal{P})$. Denote by $\mathcal{H}_{\mathscr{L}}$ the full subcategory of $\mathcal{M}$ with objects in $\mathcal{H} \cap \mathscr{L}$. Note that $\mathcal{H}_{\mathscr{L}}$ is an exact category and we denote by $D^b(\mathcal{H}_{\mathscr{L}})$ its bounded derived category.

The next definition is from [7]. Let $S_i : 1 \leq i \leq d$ be multiplicatively closed sets in R and let S be the d-tuple $S_1 \times S_2 \times \ldots S_d$.

Definition 2.2. $M \in \mathcal{M}$ is said to be S-torsion if $S_i^{-1} M = 0$ for all $i = 1, \ldots, d$. We denote the full subcategory of S-torsion modules by $\mathscr{L}_S$.

Remark 2.1. In [7], $\mathscr{L}_S$ is denoted by $S - tor$. Note that $\mathscr{L}_S = \cap \mathscr{L}_{S_i}$ and hence (or by direct verification) is a Serre subcategory of $\mathcal{M}$.

Definition 2.3. Let I be an ideal of R. $M \in \mathcal{M}$ is said to be I-torsion if $\exists n$ such that $I^n M = 0$. We denote the full subcategory of I-torsion modules by $\mathscr{L}_{V(I)}$.

In the next lemma, we relate these definitions and mention some consequences for further use. The proofs are easy and we omit them.

Lemma 2.1. *Let $\mathscr{L}$ be a Serre subcategory of $\mathcal{M}$ and I be an ideal of R.*

(1) $M \in \mathcal{M}$ *is* I*-torsion iff* $I \subseteq \sqrt{Ann(M)}$.
(2) $M \in \mathcal{M}$ *is* I*-torsion iff* M *is* $\sqrt{I}$*-torsion (justifying the notation* $\mathscr{L}_{V(I)}$*).*
(3) $\mathscr{L}_{V(I)}$ *is a Serre subcategory of* $\mathcal{M}$.
(4) *If* $I = (f_1, f_2, \dots, f_n)$ *is a generating set and* $S = S_1 \times S_2 \times \dots S_n$ *where* S_i *is the set of powers of* f_i*, then* $\mathscr{L}_S = \mathscr{L}_{V(I)}$.
(5) *If* $R/I \in \mathscr{L}$ *then every finitely generated* R/I*-module is contained in* $\mathscr{L}$.
(6) *If* $M \in \mathscr{L}$ *such that* $I = Ann(M)$ *then* $R/I \in \mathscr{L}$.

2.2. **Preliminaries from commutative algebra.** In this subsection, we start by recalling the notions of regular sequences, grade and codimension and generalizing the latter definitions to Serre subcategories. Since this material is standard, we omit proofs and refer to [5, Chapter 1] for clarifications or further reading.

Definition 2.4. Let $M \in \mathcal{M}$. A sequence $a_1, \dots, a_n$ of elements of R is called an M-regular sequence or simply an M-sequence if :

i) a_i is a non-zero divisor on $M/(a_1, \dots, a_{i-1})M$, for $i = 1, \dots, n$.
ii) $M/(a_1, \dots, a_n)M \neq 0$.

Remark 2.2. An M-sequence is an R-sequence.

Definition 2.5. Let $M \in \mathcal{M}$ and I be an ideal in R. Then the grade of I on M is defined as

$$grade(I, M) = min\{i : Ext^i_R(R/I, M) \neq 0\} \ .$$

Theorem 2.2. *Let* $M \in \mathcal{M}$ *and* I *be an ideal in* R*. If* $IM \neq M$*, then all maximal* M*-sequences in* I *have length* $= grade(I, M)$.

If $IM = M$*, then* $grade(I, M) = \infty$ *, i.e.* $Ext^i_R(R/I, M) = 0\ \forall i$.

Remark 2.3. It is customary to define $grade(I) = grade(R/I) = grade(I, R)$ for an ideal $I \subset R$. Note that $grade(I) \leq ht(I)$.

Proposition 2.3. *Let* $M \in \mathcal{M}$ *and* I *be an ideal in* R*. Then* $grade(I, M) = grade(\sqrt{I}, M)$.

Definition 2.6. Let $\mathscr{L}$ be a Serre subcategory of $\mathcal{M}$. Define

$$grade(\mathscr{L}) = min\,\{grade(I) : R/I \in \mathscr{L}\}\ , \quad codim(\mathscr{L}) = min\,\{ht(\mathfrak{p}) : R/\mathfrak{p} \in \mathscr{L}\}\,.$$

Remark 2.4. Clearly $codim(\mathscr{L}_{V(I)}) = ht(I)$. By proposition 2.3 and the definitions, $grade(\mathscr{L}_{V(I)}) = grade(I)$. Further, by remark 2.3, $grade(\mathscr{L}) \leq codim(\mathscr{L})$.

Lemma 2.4. *Suppose* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence in* $\mathcal{M}$ *with* $grade(I, M) \geq g$ *and* $grade(I, M') \geq g'$*. Then* $grade(I, M'') \geq min\{g' - 1, g\}$.

We now state the relation between $grade(\mathscr{L})$ and complexes in $Ch^b_{\mathscr{L}}(\mathcal{P})$ that we will use subsequently. While the results are not new in principle, they may not be available in this form elsewhere and hence we provide details.

Lemma 2.5. *Let* $P. \in Ch^b(\mathcal{P})$ *such that* $I \subseteq \cap_i Ann(H_i(P.))$ *and* $grade(I) > 0$*. Let* $B_i = \partial_{i+1}(P_{i+1})$ *and* $Z_i = ker(\partial_i)$*. If* $H_i \neq 0$ *then* $grade(I, B_i) = 1$.

Proof. The short exact sequence $0 \to B_i \to Z_i \to H_i \to 0$ yields the long exact sequence $0 \to Ext^0(R/I, B_i) \to Ext^0(R/I, Z_i) \to$

$$Ext^0(R/I, H_i) \to Ext^1(R/I, B_i) \to Ext^1(R/I, Z_i) \to \dots$$

Since $H_i \neq 0$ and $IH_i = 0$, $Ext^0(R/I, H_i) \simeq Hom(R/I, H_i) \simeq H_i \neq 0$. Since Z_i is a non-zero submodule of a free module and $grade(I) = g > 0$, we obtain that

$grade(I, Z_i) \geq 1$ and hence $Ext^0(R/I, Z_i) = 0$. Therefore, $Ext^1(R/I, B_i) \neq 0$ and $Ext^0(R/I, B_i) = 0$ and hence $grade(I, B_i) = 1$. □

Theorem 2.6. *Let $P. \in Ch^{[0,l]}(\mathcal{P})$ such that the homology modules are annihilated by an ideal I with $0 < grade(I) = g$. Then $H_i(P.) = 0$ for $i = l - g + 1, \ldots, l$.*

Proof. Note that $B_l = 0$ and hence $grade(I, B_i) = \infty$. Since $g > 0$, we obtain from lemma 2.5 that $H_l = Z_l = 0$. If $g = 1$, we are done. So assume $g \geq 2$.

$$\text{Claim : } grade(I, B_{l-k}) \geq g - k + 1, H_{l-k} = 0 \text{ for } 0 \leq k \leq g - 1.$$

We have already shown the claim is true for $k = 0$. We will prove the claim by induction. Now assume the claim for some $k < g - 1$, i.e. $grade(I, B_{l-k}) \geq g - k + 1$ and $H_{l-k} = 0$. Hence $Z_{l-k} = B_{l-k}$ yielding the short exact sequence $0 \to B_{l-k} \to P_{l-k} \to B_{l-k-1} \to 0$. Also note that $grade(I, B_{l-k}) - 1 \geq g - k \geq 2$ and $grade(I, P_{l-k}) \geq g \geq 2$. Hence by lemma 2.4 , we get that

$$grade(I, B_{l-k-1}) \geq min\left(grade(I, B_{l-k}) - 1, grade(I, P_{l-k})\right) \geq g - k \geq 2.$$

since otherwise we will get a contradiction to . The first inequality yields the first part of the induction statement to be proven and the second inequality along with lemma 2.5 yields the second part of the induction statement to be proven. Thus $H_k = 0$ for $k = l - g + 1, \ldots, l$ which completes the proof. □

Corollary 2.7. *Let $P. \in Ch^{[0,l]}_{\mathscr{L}}(\mathcal{P})$ such that $grade(\mathscr{L}) = g > 0$. Then $H_i(P.) = 0$ for $i = l - g + 1, \ldots, l$.*

Proof. Let $I = Ann(\oplus_{i=1}^{l} H_i(P.))$ and note that $R/I \in \mathscr{L}$ by 2.1[6]. Hence, $grade(I) \geq g$ by definition 2.6 and the first statement now follows by a direct application of theorem 2.6. □

As an immediate consequence, we obtain the following corollary which is the form we require for use in section 6.

Corollary 2.8. *Let $P. \in Ch^{b}_{\mathscr{L}}(\mathcal{P})$ such that $grade(\mathscr{L}) = g > 0$.*

(1) If $P. \in Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})$ is not exact, then $amplitude(P.) \geq g$.
(2) If $amplitude(P.) = g$ then $P.$ is a projective resolution of $H_{min(P.)}(P.)$ and in particular $H_{min(P.)}(P.) \in \mathcal{H} \cap \mathscr{L}$.

We now recall the new intersection theorem which gives even better bounds on the amplitude when the Krull dimension is finite.

Theorem 2.9. *(New Intersection Theorem ; [18], [19]) Let R be a local ring of finite Krull dimension and $P. \in Ch^b(\mathcal{P})$ be such that all its homology modules are of finite length. If $P.$ is not exact, then $amplitude(P.) \geq dim(R)$.*

Lemma 2.10. *Let $\mathscr{L}$ be a Serre subcategory of $\mathcal{M}$ with $codim(\mathscr{L}) < \infty$ and $P. \in Ch^b_{\mathscr{L}}(\mathcal{P})$. If $P.$ is not exact, then $amplitude(P.) \geq codim(\mathscr{L})$.*

Proof. Let $I = Ann\left(\left(\bigoplus H_i(P.)\right)\right)$ and consider a minimal prime ideal $\mathfrak{p}$ of I. Then $R/I \in \mathscr{L}$ by lemma 2.1[6] and hence $R/\mathfrak{p} \in \mathscr{L}$. Consider the complex $P._{\mathfrak{p}}$ and note that

$$Ann\left(\bigoplus\left(H_i\left(P._{\mathfrak{p}}\right)\right)\right) = Ann\left(\bigoplus\left(H_i\left(P.\right)\right)\right)_{\mathfrak{p}}.$$

Moreover $Supp\left(\bigoplus H_i\left(P._{\mathfrak{p}}\right)\right)$ is the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$. Hence $P._{\mathfrak{p}}$ consists of finite rank free $R_{\mathfrak{p}}$ modules with non-zero homology of finite length. Thus, by theorem 2.9 and definition 2.6,

$$amplitude\,(P.) = amplitude\,(P._{\cdot\mathfrak{p}}) \geq dim(R_{\mathfrak{p}}) = ht(\mathfrak{p}) \geq codim(\mathscr{L}).$$

□

We end this subsection by recalling the definition of the arithmetic rank.

Definition 2.7. Let $I \subseteq R$ be an ideal. We define the arithmetic rank of I (denoted by $ara(I)$) as :

$$ara(I) := min\{s|\sqrt{I} = \sqrt{(a_1, \ldots a_s)} \text{ for some } a_1, \ldots a_s \in R\}$$

2.3. **Preliminaries on K_0 and the invariants.**

Definition 2.8. Let $\mathscr{A}$ be an exact category (resp. triangulated category). We define $K_0(\mathscr{A})$ to be the free abelian group on isomorphism classes of objects of the category modulo the following relation : $[p] = [p_1] + [p_2]$ if there is a short exact sequence $0 \to p_1 \to p \to p_2 \to 0$ (resp. exact triangle $p_1 \to p \to p_2 \to \Sigma p_1$) in $\mathscr{A}$.

For a Serre subcategory $\mathscr{L}$, recall that $K_0(R \text{ on } \mathscr{L})$ is defined as $K_0(D^b_{\mathscr{L}}(\mathcal{P}))$ as mentioned earlier. We define the n^{th} shift of a complex.

Definition 2.9. Let $P.$ be a complex. Define the complex (called its n^{th} shift) $\Sigma^n P.$ by $(\Sigma^n P.)_l = P_{l-n}$ and differentials $\partial_l^{\Sigma^n P.} = (-1)^n \partial_{l-n}^{P.}$.

We state a lemma connecting the classes of a complex and its shifts. The proof uses the usual cone construction and we omit it.

Lemma 2.11. *Let $P., Q. \in Ch^b_{\mathscr{L}}(\mathcal{P})$.*

(1) $[(\Sigma^n Q.)] = (-1)^n [Q.] \in K_0(R \text{ on } \mathscr{L})$.

(2) If $\Sigma^n Q., Q. \in Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$, *then* $[(\Sigma^n Q.)] = (-1)^n [Q.] \in \dfrac{K_0\left(Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$.

(3) If $P., Q. \in Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})$ *and* $P. \xrightarrow{f} Q.$ *is a quasi-isomorphism, then*

$$[P.] = [Q.] \in \frac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}.$$

We give an alternate description of $K_0(R \text{ on } \mathscr{L})$ by refining the set of relations we need to go modulo in the definition.

Lemma 2.12. *$K_0(R$ on $\mathscr{L})$ can be obtained by going modulo the following relations on the free abelian group on isomorphism classes of $Ch^b_{\mathscr{L}}(\mathcal{P})$.*

(1) $[P.] = 0$ *if $P.$ is exact.*

(2) $[P.] = [P.'] + [P.'']$ *if there is a short exact sequence* $0 \to P.' \to P. \to P.'' \to 0$ *in $Ch^b(\mathcal{P})$.*

Proof. We will show that the relations in definition 2.8 can be obtained from these ones. Note first that by the same proof as in lemma 2.11, $[(\Sigma^n Q.)] = (-1)^n [Q.]$ Now if $P. \xrightarrow{f} P.'$ is a quasi-isomorphism, then using that $Cone(f)$ is exact and the first relation, we get that $[P.] + [\Sigma P.'] = 0$. Then using that shifting changes sign, we get that $[P.] = [P.']$.

Now suppose $P.$ and $P.'$ are isomorphic in $D^b_{\mathscr{L}}(\mathcal{P})$. Then there is a zig-zag of quasi-isomorphisms linking $P.$ and $P.'$ and hence successive use of the above shows that $[P.] = [P.']$. Thus, we get that isomorphism classes in $D^b_{\mathscr{L}}(\mathcal{P})$ have the same class.

Now assume that $Q.' \to Q. \to Q.'' \to \Sigma Q.'$ is an exact triangle in $D^b_{\mathscr{L}}(\mathcal{P})$ and we want to prove that $[Q.] = [Q.'] + [Q.'']$ using the above relations. By definition, exact triangles in $D^b_{\mathscr{L}}(\mathcal{P})$ are precisely those which are isomorphic (as triangles) to triangles obtained via tbe standard cone construction, so the above triangle is isomorphic to a triangle of the form $P.' \xrightarrow{f} P. \to Cone(f) \to \Sigma P.'$. The isomorphism (of triangles) induces isomorphisms $P.' \simeq Q.', P. \simeq Q., Cone(f) \simeq Q.''$ in $D^b_{\mathscr{L}}(\mathcal{P})$ and by the previous paragraph, it is thus enough to prove that $[P.] = [P.'] + [Cone(f)]$. Since $0 \to P. \to Cone(f) \to \Sigma P.' \to 0$ is a short exact sequence in $Ch^b(\mathcal{P})$, we get from the second relation that $[P.] + [\Sigma P.'] = [Cone(f)]$. Using 2.11, we have thus shown that the relations in definition 2.8 can be obtained from these ones.

Since isomorphisms in $Ch^b_{\mathscr{L}}(\mathcal{P})$ induce isomorphisms in $D^b_{\mathscr{L}}(\mathcal{P})$, exact complexes are isomorphic to 0. in $D^b_{\mathscr{L}}(\mathcal{P})$ and short exact sequences give rise to exact triangles in $D^b_{\mathscr{L}}(\mathcal{P})$, the above relations can clearly be obtained from the ones in definition 2.8. This proves the lemma. □

Corollary 2.13. $\dfrac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \textit{ is exact } \rangle} \xrightarrow{\sim} \dfrac{K_0\left(Ch^{b}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \textit{ is exact } \rangle} \xrightarrow{\sim} K_0(R \textit{ on } \mathscr{L}).$

Proof. The first isomorphism follows from lemma 2.11 allowing shifting upto sign, and the second one follows from lemma 2.12 above. □

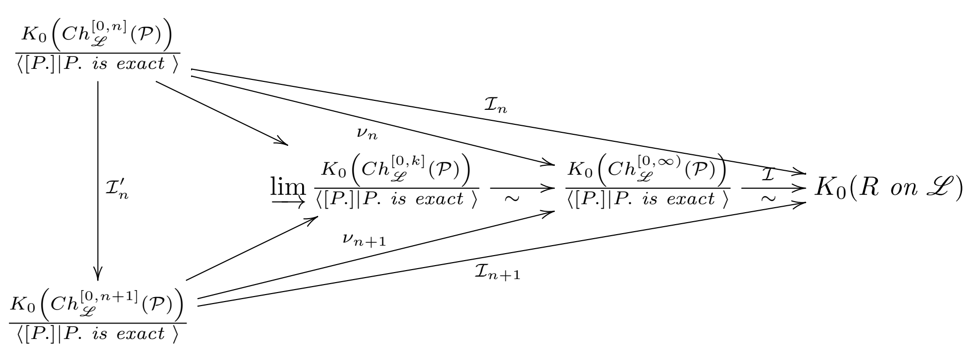
Let $\mathcal{I}$ be the composite map in the previous corollary. Note that there are natural maps $\mathcal{I}'_n : \frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \to \frac{K_0\left(Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$ and $\frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \xrightarrow{\nu_n} \frac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$ both obtained by sending $[Q.]$ to $[Q.]$. Define $\mathcal{I}_n = \mathcal{I} \circ \nu_n$.

Lemma 2.14. $\varinjlim \dfrac{K_0\left(Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \textit{ is exact } \rangle} \xrightarrow{\sim} \dfrac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \textit{ is exact } \rangle}.$

Proof. Clearly $\nu_n = \nu_{n+1} \circ \mathcal{I}'_n$ producing the required homomorphism. Surjectivity and injectivity then follow since the checking involves finitely many complexes each of finite amplitude in $Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})$ and hence can as well be done in some $Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$. □

As an immediate consequence, we obtain the following corollary, a special case of which is a key ingredient in the proofs in [7] and we will also use subsequently.

Corollary 2.15. *We have the following commutative diagram :*



Remark 2.5. In [7], the homology support is generically denoted by a condition #. If we let $\mathscr{L}_\#$ to be the full subcategory of modules satisfying #, then the corresponding group $\frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}_\#}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$ is denoted in [7] by $K_0(P_n(\#))$ and the group $K_0(R \text{ on } \mathscr{L}_\#)$ is denoted by $K_0(P(\#))$.

We define the invariants (mentioned in the introduction) of study in this article.

Definition 2.10. Define $\alpha_s(\mathscr{L})$ $(/\alpha_i(\mathscr{L})/\alpha(\mathscr{L}))$ to be the smallest n for which $\mathcal{I}'_r$ is surjective (resp. injective/isomorphic) for all $r \geq n$.

Remark 2.6. Note that by corollary 2.15 and the properties of direct limits, $\mathcal{I}'_r$ is injective/isomorphic for all $r \geq n$ iff ν_r is injective/isomorphic for all $r \geq n$ iff $\mathcal{I}_r$ is injective/isomorphic for all $r \geq n$.

However, $\mathcal{I}'_r$ is surjective for all $r \geq n$ is only sufficient to conclude ν_r (and hence $\mathcal{I}_r$) is surjective for all $r \geq n$.

Also, clearly $\alpha(\mathscr{L}) = \max\{\alpha_i(\mathscr{L}), \alpha_s(\mathscr{L})\}$.

We now state a lemma which will allow bounds for $\alpha(\mathscr{L})$ for a Serre subcategory $\mathscr{L}$ in terms of a "cover" by smaller Serre subcategories.

Lemma 2.16. *Let $\mathscr{L}, \{\mathscr{L}_\lambda\}$ be Serre subcategories of $\mathcal{M}$ such that $\mathscr{L} = \cup_\lambda \mathscr{L}_\lambda$. Then* $\alpha_s(\mathscr{L}) \leq \max_\lambda \alpha_s(\mathscr{L}_\lambda)$, $\alpha_i(\mathscr{L}) \leq \max_\lambda \alpha_i(\mathscr{L}_\lambda)$, $\alpha(\mathscr{L}) \leq \max_\lambda \alpha(\mathscr{L}_\lambda)$.

Proof. Clearly there is a natural map $\varinjlim K_0\left(Ch^{[0,k]}_{\mathscr{L}_\lambda}(\mathcal{P})\right) \xrightarrow{\Psi_k} K_0\left(Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})\right)$ where $k = 0, 1, 2, \ldots, \infty$. Suppose $P. \in Ch^b_{\mathscr{L}}(\mathcal{P})$. Let $I = Ann(\oplus_{i=-\infty}^{\infty} H_i(P.))$. Then by lemma 2.1, $R/I \in \mathscr{L}$ and hence there exists λ such that $R/I \in \mathscr{L}_\lambda$ and so again using lemma 2.1, $P. \in Ch^b_{\mathscr{L}_\lambda}(\mathcal{P})$. Hence $Ch^b_{\mathscr{L}}(\mathcal{P}) = \cup_\lambda Ch^b_{\mathscr{L}_\lambda}(\mathcal{P})$. In particular this implies that Ψ_k is surjective.

For injectivity, let $\tau = [P.] - [Q.] \in \varinjlim K_0\left(Ch^{[0,k]}_{\mathscr{L}_\lambda}(\mathcal{P})\right)$ such that $\Psi_k(\tau) = 0$. Then there is an equation of the form $[P.] + \sum([P'_i.] + [Q'_i.]) = [Q.] + \sum([P'_i. \oplus Q'_i.])$ in the free abelian group on isomorphism classes in $Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$. Since the equation involves finitely many complexes, each of finite amplitude in $Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$. By considering the annihilator of the sum of all the homologies involved, a similar proof as in the previous paragraph will show that the equation will hold in $Ch^{[0,k]}_{\mathscr{L}_\lambda}(\mathcal{P})$ for some λ and hence $\tau = 0$. Hence, Ψ_k is injective.

Going modulo the subgroup generated by classes of exact complexes, we get the following commutative diagram :

$$\begin{array}{ccccc}
\frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}_\lambda}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} & \longrightarrow & \varinjlim_\lambda \frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} & \xrightarrow[\sim]{\Psi_n} & \frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \\
\downarrow \mathcal{I}'_n(\lambda) & & \downarrow & & \downarrow \mathcal{I}'_n \\
\frac{K_0\left(Ch^{[0,n+1]}_{\mathscr{L}_\lambda}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} & \longrightarrow & \varinjlim_\lambda \frac{K_0\left(Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} & \xrightarrow[\sim]{\Psi_{n+1}} & \frac{K_0\left(Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}
\end{array}$$

We prove the required statement for α_s. Let $n \geq \max_\lambda \alpha_s(\mathscr{L}_\lambda)$. Then $\mathcal{I}'_n(\lambda)$ is surjective for each λ and each n and hence by properties of the direct limit, the middle vertical arrow is also an isomorphism for each n. Therefore $\mathcal{I}'_n$ is an isomorphism for all n and hence we get that $n \geq \alpha_s(\mathscr{L})$. The statements for α_i and

α follow by replacing surjective in the above paragraph by injecive and isomorphic respectively. This completes the proof. □

We end this subsection by obtaining a lower bound on $\alpha_s(\mathscr{L})$.

Corollary 2.17. *If $\mathscr{L} \neq \{0\}$, then $grade(\mathscr{L}) \leq codim(\mathscr{L}) \leq \alpha_s(\mathscr{L})$.*

Proof. For $k < codim(\mathscr{L})$, by lemma 2.10, $\frac{K_0\left(Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} = 0$. If $\mathcal{I}'_n$ is surjective $\forall n \geq k$, then $K_0(R \text{ on } \mathscr{L}) = 0$ implying $\mathscr{L} = \{0\}$. Hence, $codim(\mathscr{L}) \leq \alpha_s(\mathscr{L})$. □

3. Reducing the amplitude of complexes

In this section, we introduce the main technique used to reduce the amplitude of complexes and hence providing a bound on $\alpha(\mathscr{L})$ and hence $\alpha_s(\mathscr{L})$. Indeed, other than a few special cases, this is the only way we know of obtaining bounds. While the idea is motivated by the proof of the main theorem in [7], it is to be pointed out that in that article the authors reduce the complex from the top rather than the bottom which is possible due to a very special Koszul complex that they explicitly define. It appears though that this makes the fundamental property of projective modules much harder to use (kernels of surjections of projectives are projective, but cokernels of injections need not be) and it seems to be more natural (and easier) to make the reduction from below rather than above. Indeed this idea has been used in [20] and the proofs use ideas from the main constructions of both these articles.

Definition-Remark 3.1. Suppose $P. \in Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ and there exists $(Q., u)$ where $Q. \in Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})$ and $Q. \xrightarrow{u} P.$ with u_0 a surjection. We obtain the following short exact sequence of complexes

$$0 \to P. \to cone(u). \to \Sigma Q. \to 0. \tag{1}$$

Since $u_0 : Q_0 \twoheadrightarrow P_0$, there is a splitting $s : P_0 \to Q_0$. Let t be the induced (left splitting) map from $Q_0 \twoheadrightarrow ker(u_0)$. Recall that if $i : ker(u_0) \to Q_0$ is the natural inclusion, $i \circ t = 1 - s \circ u_0$ and $t \circ i = 1$.

Define $B.$ concentrated within degrees 0 and 1 as :

$$B. : \quad \dots 0 \longrightarrow P_0 \xrightarrow{1} P_0 \longrightarrow 0 \dots$$

and define a morphism $a : B. \to cone(u).$ by :

$$B. \xrightarrow{a} cone(u).$$

$$\begin{array}{cccc} 1 & P_0 & \xrightarrow{\begin{pmatrix} s \\ 0 \end{pmatrix}} & Q_0 \oplus P_1 \\ & \downarrow 1 & & \downarrow \begin{pmatrix} -u_0 & \partial_1^P \end{pmatrix} \\ 0 & P_0 & \xrightarrow[-1]{} & P_0 \end{array}$$

Clearly a is injective and $coker(a)_0 = 0$. Define $D.(Q., u, s)$ as the complex

$$\dots 0 \longrightarrow cone(u)_{n+1} \xrightarrow{\partial} \dots \xrightarrow{\partial} cone(u)_2 \xrightarrow{\eta(Q.,u,s)} ker(u_0) \oplus P_1 \longrightarrow 0 \dots$$

$$n \qquad\qquad\qquad 1 \qquad\qquad\qquad 0$$

where the differentials ∂ are the same as in $cone(u)$ and

$$\eta(Q., u, s) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -d_1^Q & 0 \\ -u_1 & d_2^P \end{bmatrix}.$$

Then clearly, $D. \xrightarrow{-1} \Sigma^{-1} coker(a).$ is an isomorphism and hence we have a short exact sequence

$$0 \to B. \xrightarrow{a} cone(u) \to D.(Q., u, s) \to 0. \tag{2}$$

Note that $Q., D.(Q., u, s), \in Ch_{\mathscr{L}}^{[0,n]}(\mathcal{P})$ and in $\frac{K_0\left(Ch_{\mathscr{L}}^{[0,n+1]}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$

$$\begin{aligned} [P.] &= [cone(u)] + [Q.] && \text{(using (1))} \\ &= [\Sigma D.(Q., u, s)] + [B.] + [Q.] && \text{(using (2))} \\ &= [Q.] - [D.(Q., u, s)]. \end{aligned}$$

We define $(Q., u)$ as a reducer for $P.$

As an immediate consequence, we obtain :

Corollary 3.1. *If every $P. \in Ch_{\mathscr{L}}^{[0,n+1]}(\mathcal{P})$ has a reducer, then $\mathcal{I}'_n$ is surjective.*

We now show that under the hypothesis of Corollary 3.1, $\mathcal{I}'_n$ is an isomorphism by constructing an explicit inverse to $\mathcal{I}'_n$. The key is to observe that the classes $[Q.] - [D.(Q., u, s)]$ are independent of all choices.

Lemma 3.2. *With the notations of definition-remark 3.1, let s, s' be sections of $u_0 : Q_0 \twoheadrightarrow P_0$. Then $[D.(Q., u, s)] = [D.(Q., u, s')]$ in $\dfrac{K_0\left(Ch_{\mathscr{L}}^{[0,n]}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$*

Proof. Since $u_0(s - s') = 0$, there is an induced map $P_0 \xrightarrow{f} ker(u_0)$ such that $i \circ f = s - s'$. Note that $i\left(-td_1^Q - fd_1^P u_1\right) = -itd_1^Q - sd_1^P u_1 + s'd_1^P u_1 = -itd_1^Q - su_0 d_1^Q + s'u_0 d_1^Q = -d_1^Q + s'u_0 d_1^Q = -it'd_1^Q$ and hence $-td_1^Q - fd_1^P u_1 = -t'd_1^Q$. Using this, we obtain the following matrix identity :

$$\begin{aligned} \begin{bmatrix} 1 & fd_1^P \\ 0 & 1 \end{bmatrix} \eta(Q., u, s) &= \begin{bmatrix} 1 & fd_1^P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -d_1^Q & 0 \\ -u_1 & d_2^P \end{bmatrix} = \begin{bmatrix} t & fd_1^P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -d_1^Q & 0 \\ -u_1 & d_2^P \end{bmatrix} \\ &= \begin{bmatrix} -td_1^Q - fd_1^P u_1 & 0 \\ -u_1 & d_2^P \end{bmatrix} = \begin{bmatrix} -t'd_1^Q & 0 \\ -u_1 & d_2^P \end{bmatrix} \\ &= \begin{bmatrix} t' & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -d_1^Q & 0 \\ -u_1 & d_2^P \end{bmatrix} = \eta(Q., u, s') \end{aligned}$$

This yields an explicit isomorphism $D.(Q., u, s) \xrightarrow{\sim} D.(Q., u, s')$ given by :

$$\begin{array}{ccccccc} cone(u)_n & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & Q_1 \oplus P_2 & \xrightarrow{\eta(s)} & ker(u_0) \oplus P_1 \\ \downarrow 1 & & \downarrow 1 & & \downarrow \begin{bmatrix} 1 & fd_P^1 \\ 0 & 1 \end{bmatrix} \\ cone(u)_n & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & Q_1 \oplus P_2 & \xrightarrow{\eta(s')} & ker(u_0) \oplus P_1 \end{array}$$

∎

Definition 3.2. Let $P. \in Ch_{\mathscr{L}}^{[0,n+1]}(\mathcal{P})$ and $(Q., u)$ be a reducer. Define $\mathcal{J}_n(Q., u) = [Q.] - [D.(Q., u, s)]$ in $\frac{K_0\left(Ch_{\mathscr{L}}^{[0,n]}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$ where s is any section of u_0. By lemma 3.2, this is well-defined.

Proposition 3.3. *Let $P. \in Ch_{\mathscr{L}}^{[0,n+1]}(\mathcal{P})$ and $(Q., u)$ and $(Q.', u')$ be two reducers of P. Then $\mathcal{J}_n(Q., u) = \mathcal{J}_n(Q.', u')$.*

Proof. definition-remark 3.1 Choose sections s and s' for u_0 and U_0' respectively. Note that $(Q. \oplus Q.', [u, u'])$ is also a reducer for $P.$ and consider its section $\begin{pmatrix} s \\ 0 \end{pmatrix}$. We will show :

$$\textit{Claim} : [Q.'] + [D.(Q., u, s)] = \left[D.\left(Q. \oplus Q.', [u, u'], \begin{pmatrix} s \\ 0 \end{pmatrix}\right)\right].$$

Assuming the claim, we obtain

$$\begin{aligned}
\mathcal{J}_n(Q., u) &= [Q.] - [D.(Q., u, s)] = [Q.] + [Q.'] - ([Q.'] + [D.(Q., u, s)]) \\
&= [Q. \oplus Q.'] - \left[D.\left(Q. \oplus Q.', [u, u'], \begin{pmatrix} s \\ 0 \end{pmatrix}\right)\right] \quad \text{(by the claim)} \\
&= \mathcal{J}_n(Q. \oplus Q.', [u, u']).
\end{aligned}$$

By symmetry, $\mathcal{J}_n(Q.', u') = \mathcal{J}_n(Q. \oplus Q.', [u, u'])$ thus proving the proposition.

We now prove the claim. Recall the notations of definition-remark 3.1. It is clear that the injection $cone(u). \xrightarrow{i} cone([u, u'])$. fits into a commutative square

$$\begin{array}{ccccc}
0 & \longrightarrow & B. & \xrightarrow{a(s)} & cone(u). \\
 & & \Vert & a\left(\begin{pmatrix} s \\ 0 \end{pmatrix}\right) & \downarrow i \\
0 & \longrightarrow & B. & \longrightarrow & cone([u, u']).
\end{array}$$

and hence there is an induced map $\bar{i} : coker(a(s)) \to coker\left(a\left(\begin{pmatrix} s \\ 0 \end{pmatrix}\right)\right)$. By the snake lemma, $\bar{i}$ is injective and $coker(i) \xrightarrow{\sim} coker(\bar{i})$. Since $coker(i) \xrightarrow{\sim} \Sigma Q.'$, we obtain a short exact sequence

$$0 \to \Sigma^{-1} coker(a(s)) \xrightarrow{\Sigma^{-1}\bar{i}} \Sigma^{-1} coker\left(a\begin{pmatrix} s \\ 0 \end{pmatrix}\right) \to Q'. \to 0 \quad .$$

As observed in definition-remark 3.1 , $D(Q., u, s) \xrightarrow{-1} \Sigma^{-1} coker(s)$ and similarly $D.\left(Q. \oplus Q.', [u, u'], \begin{pmatrix} s \\ 0 \end{pmatrix}\right) \xrightarrow{-1} \Sigma^{-1} coker\left(\begin{pmatrix} s \\ 0 \end{pmatrix}\right)$ and hence, we obtain a short exact sequence

$$0 \to D(Q., u, s) \to D.\left(Q. \oplus Q.', [u, u'], \begin{pmatrix} s \\ 0 \end{pmatrix}\right) \to Q.' \to 0$$

from which the claim follows. □

Definition 3.3. Suppose $P. \in Ch_{\mathscr{L}}^{[0,n+1]}(\mathcal{P})$ has a reducer. Define $\mathcal{J}_n(P.) = \mathcal{J}_n(Q., u)$ for any reducer $(Q., u)$ of $P.$ The above proposition 3.3 shows that $\mathcal{J}_n(P.)$ is well-defined. Clearly, if $P. \xrightarrow{\sim} P.'$, then $\mathcal{J}_n(P.) = \mathcal{J}_n(P.')$.

Now suppose every $P. \in Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ has a reducer. Clearly, if $P. \xrightarrow{\sim} P.'$, then $\mathcal{J}_n(P.) = \mathcal{J}_n(P.')$. Hence $\mathcal{J}_n$ extends to a group homomorphism $\mathcal{J}_n$ from the free abelian group on isomorphism classes of $Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ to $\frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$.

The following lemma shows that $\mathcal{J}_n$ factors through $K_0\left(Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})\right)$.

Lemma 3.4. *Let $0 \to P.' \xrightarrow{f} P. \xrightarrow{g} P.'' \to 0$ be a short exact sequence in $Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ and suppose $P.', P.$ have reducers. Then $\mathcal{J}_n([P.] - [P.''] - [P.']) = 0$.*

Proof. Choose a splitting $P''_0 \xrightarrow{t} P_0$ of $P_0 \xrightarrow{g_0} P''_0$ and let $P_0 \xrightarrow{t'} P'_0$ be the induced (left) splitting. Choose $(Q.', u', s')$ and $(\tilde{Q}., \tilde{u}, \tilde{s})$ as reducers and sections for $P.'$ and $P.$ respectively. Define

$$Q.'' = \tilde{Q}. \quad , \quad u'' = g \circ \tilde{u} \quad , \quad s'' = \tilde{s} \circ t$$

$$Q. = Q.' \oplus Q.'' \quad , \quad u = \begin{pmatrix} f \circ u' & \tilde{u} \end{pmatrix} \quad , \quad s = \begin{pmatrix} s' \circ t' \\ \tilde{s} \circ t \circ g_0 \end{pmatrix}.$$

Clearly, $(Q.'', u'')$ and $(Q., u)$ are reducers for $P.'$ and $P.$ respectively. Note that $u''_0 s'' = g_0 \tilde{u}_0 \tilde{s} t = g_0 t = 1$ and

$$u_0 s = \begin{pmatrix} f_0 \circ u'_0 & \tilde{u}_0 \end{pmatrix} \begin{pmatrix} s' \circ t' \\ \tilde{s} \circ t \circ g_0 \end{pmatrix} = f_0 u'_0 s' t' + \tilde{u}_0 \tilde{s} t g_0 = f_0 t' + t g_0 = 1.$$

Hence s'' and s are sections for u''_0 and u_0 respectively. By construction,we have a commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q.' & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & Q. & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & Q.'' & \longrightarrow & 0 \\ & & \Big\downarrow u' & & \Big\downarrow u & & \Big\downarrow u'' & & \\ 0 & \longrightarrow & P.' & \xrightarrow{f} & P. & \xrightarrow{g} & P.'' & \longrightarrow & 0 \end{array} \tag{3}$$

Since the sections satisfy $sf_0 = \begin{pmatrix} s' \circ t' \\ \tilde{s} \circ t \circ g_0 \end{pmatrix} f_0 = \begin{pmatrix} s' \circ t' \circ f_0 \\ \tilde{s} \circ t \circ g_0 \circ f_0 \end{pmatrix} = \begin{pmatrix} s' \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} s'$ and $\begin{pmatrix} 0 & 1 \end{pmatrix} s = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s' \circ t' \\ \tilde{s} \circ t \circ g_0 \end{pmatrix} = \tilde{s} \circ t \circ g_0 = s'' \circ g_0$, we have a diagram at degree 0 where the sections also commute :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q'_0 & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & Q_0 & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & Q''_0 & \longrightarrow & 0 \\ & & s' \Big\uparrow \Big\downarrow u'_0 & & s \Big\uparrow \Big\downarrow u_0 & & s'' \Big\uparrow \Big\downarrow u''_0 & & \\ 0 & \longrightarrow & P'_0 & \underset{t'}{\overset{f_0}{\rightleftarrows}} & P_0 & \underset{t}{\overset{g_0}{\rightleftarrows}} & P''_0 & \longrightarrow & 0 \end{array} \tag{4}$$

As a consequence of diagram (3), we obtain a short exact sequence of cones

$$0 \to cone(u'). \xrightarrow{F} cone(u). \xrightarrow{G} cone(u''). \to 0 \quad .$$

Recalling the notations of definition-remark 3.1 , f_0 and g_0 induce a short exact sequence $0 \to B.' \xrightarrow{\tilde{f}_0} B. \xrightarrow{\tilde{g}_0} B.'' \to 0$. Using the compatibility of sections as in diagram (4) , we obtain the following commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B.' & \xrightarrow{\tilde{f}_0} & B. & \xrightarrow{\tilde{g}_0} & B.'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a(s')} & & \downarrow{\scriptstyle a(s)} & & \downarrow{\scriptstyle a(s'')} & & \\ 0 & \longrightarrow & cone(u'). & \xrightarrow{F} & cone(u). & \xrightarrow{G} & cone(u''). & \longrightarrow & 0 \end{array} .$$

Since the vertical arrows are injective, the cokernels then form an exact sequence, and applying Σ^{-1} and then the isomorphism -1 finally yields the exact sequence

$$0 \to D.(Q.', u', s') \to D.(Q., u, s) \to D.(Q.'', u'', s'') \to 0 \quad .$$

The lemma now follows since

$$\begin{aligned} \mathcal{J}_n([P.]) - \mathcal{J}_n([P.'']) - \mathcal{J}_n([P.']) &= [Q.] - [D.(Q., u, s)] - ([Q.''] - [D.(Q.'', u'', s'')]) \\ &\qquad - ([Q.'] - [D.(Q.', u', s')]) \\ &= ([Q.] - [Q.''] - [Q.']) \\ &\qquad - ([D.(Q., u, s)] - [D.(Q.'', u'', s'')] - [D.(Q.', u', s')]) \\ &= 0. \end{aligned}$$

□

Hence if every $P. \in Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ has a reducer, $\mathcal{J}_n$ induces a group homomorphism $K_0\left(Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})\right) \to \frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$. We will abuse notation and continue calling it $\mathcal{J}_n$. Note that if $P. \in Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ is exact, the good truncation $Q. = \tau_{\leq n} P.$ with the natural map (which we call u) is a reducer for $P.$ and the only possible section is the identity map. Since $P., Q.$ are exact, so is $cone(u).$ and hence so is $D.(Q., u, 1)$. Hence, $\mathcal{J}_n([P.]) = 0$. Thus, we have proved :

Theorem 3.5. *Let $\mathscr{L} \subset \mathcal{M}$ be a Serre subcategory. If every $P. \in Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$ has a reducer, then $\frac{K_0\left(Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \xrightarrow{\mathcal{J}_n} \frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$ is inverse to $\mathcal{I}'_n$. In particular, $\mathcal{I}'_n$ is an isomorphism.*

Definition 3.4. Let $\mathscr{L} \subset \mathcal{M}$ be a Serre subcategory. Define

$$\begin{aligned} \beta(\mathscr{L}) :=& min \left\{ m \in \mathbb{N} \;\middle|\; \forall n > m \text{ every } P. \in Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P}) \text{ has a reducer.} \right\} \\ =& min \left\{ m \in \mathbb{N} \;\middle|\; \forall n > m \text{ every } P. \in Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P}) \text{ has a reducer in } Ch^{[0,m]}_{\mathscr{L}}(\mathcal{P}). \right\} \end{aligned}$$

As an immediate consequence of theorem 3.5 , we obtain :

Corollary 3.6. *Let $\mathscr{L} \subset \mathcal{M}$ be a Serre subcategory. Then $\alpha(\mathscr{L}) \leq \beta(\mathscr{L})$.*

We end the section with a lemma for $\beta(\mathscr{L})$ similar to lemma 2.16.

Lemma 3.7. *Let $\mathscr{L}, \{\mathscr{L}_\lambda\}$ be Serre subcategories of $\mathcal{M}$ such that $\mathscr{L} = \cup_\lambda \mathscr{L}_\lambda$. Then $\beta(\mathscr{L}) \leq \max_\lambda \beta(\mathscr{L}_\lambda)$.*

Proof. Let $n \geq \max_{\lambda} \beta(\mathscr{L}_\lambda)$ and $P. \in Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})$. Then $\oplus_i Ann(H_i(P.)) \in \mathscr{L}$. Hence, $\oplus_i Ann(H_i(P.)) \in \mathscr{L}_\lambda$ for some λ and hence $P. \in Ch^{[0,n+1]}_{\mathscr{L}_\lambda}(\mathcal{P})$. Hence, there is a reducer $(Q., u)$ for $P.$ where $Q. \in Ch^{[0,n]}_{\mathscr{L}_\lambda}(\mathcal{P})$. Hence, $(Q., u)$ is a reducer for $P.$ in $Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})$ thus proving the lemma. □

4. Condition (**) and the arithmetic rank

In this section, we define an invariant which we call the arithmetic rank, show that this name is justified and compute it in a few cases. At the end of the section, using the main results from [7], we show that it naturally bounds $\beta(\mathscr{L})$ (and hence $\alpha(\mathscr{L})$ and $\alpha_s(\mathscr{L})$).

Definition 4.1. Let $\mathscr{L}$ be a non-zero Serre subcategory of $\mathcal{M}$. We say that $\mathscr{L}$ satisfies $(**)_r$ if the following holds :

$$\text{Whenever } M \in \mathscr{L}\ ,\ \exists f_1, \ldots, f_r \in Ann(M) \text{ such that } \tfrac{R}{(f_1, \ldots, f_r)} \in \mathscr{L}.$$

Define $ara(\mathscr{L}) = min\{r | \mathscr{L} \text{ satisfies } (**)_r\}$.

Define the arithmetic rank of the Serre subcategory $\{0\}$ to be ∞.

We show that this definition matches with the usual one for ideals.

Lemma 4.1. *$ara(\mathscr{L}_{V(I)}) = ara(I)$.*

Proof. Let us denote $\mathscr{L}_{V(I)}$ by $\mathscr{L}$ and $ara(I)$ to be a. Then $\exists g_1, \ldots, g_a$ such that $\sqrt{I} = \sqrt{(g_1, \ldots, g_a)}$. Note that $M \in \mathscr{L} \Leftrightarrow Supp(M) \subseteq V(I) \Leftrightarrow V(Ann(M)) \subseteq V(I) \Leftrightarrow \sqrt{I} \subseteq \sqrt{Ann(M)} \Leftrightarrow \sqrt{(g_1, \ldots, g_a)} \subseteq \sqrt{Ann(M)}$. Hence $g_1^{\gamma_1}, \ldots, g_a^{\gamma_a} \in Ann(M)$ and clearly $\frac{R}{(g_1^{\gamma_1}, \ldots, g_a^{\gamma_a})} \in \mathscr{L}$. Hence $\mathscr{L}$ satisfies $(**)_a$ and so $ara(\mathscr{L}) \leq a$.

Now suppose $\mathscr{L}$ satisfies $(**)_{a-1}$. With $M = R/I$ in the above equivalences, we obtain $f_1, \ldots, f_{a-1} \in Ann(R/I)$ such that $\frac{R}{(f_1, \ldots, f_{a-1})} \in \mathscr{L}$.

$$f_1, \ldots, f_{a-1} \in Ann(R/I) = I \quad\Rightarrow\quad V(f_1, \ldots, f_{a-1}) \supseteq V(I) \quad \text{and}$$
$$\frac{R}{(f_1, \ldots, f_{a-1})} \in \mathscr{L} \quad\Rightarrow\quad V(f_1, \ldots, f_{a-1}) \subseteq V(I).$$

Hence $V((f_1, \ldots, f_{a-1})) = V(I)$ which implies $\sqrt{(f_1, \ldots, f_{a-1})} = \sqrt{I}$, contradicting that $ara(I) = a$. Hence $ara(\mathscr{L}_{V(I)}) = ara(I)$. □

Lemma 4.2. *$ara(\mathscr{L}) = 0$ iff $\mathscr{L} = R - mod$.*

Proof. $ara(\mathscr{L}) = 0$ implies $\mathscr{L}$ satisfies $(**)_0$. Hence $R \in \mathscr{L}$ and therefore by lemma 2.1(1), $R{-}mod \subseteq \mathscr{L}$ which implies $\mathscr{L} = R{-}mod$. Conversely, $\mathscr{L} = R{-}mod$ implies $R \in \mathscr{L}$. Hence $\mathscr{L}$ satisfies $(**)_0$ and so $ara(\mathscr{L}) = 0$. □

Lemma 4.3. *Let $S = S_1 \times S_2 \times \ldots S_d$ be a d-tuple of multiplicatively closed sets. Then $ara(\mathscr{L}_S) \leq d$. Furthermore, if $d = 1$ and $0 \notin S$, then $ara(\mathscr{L}_S) = 1$.*

Proof. Let $M \in \mathscr{L}_S$. Hence $M \in \mathscr{L}_{S_i}$ for all $i = 1, \ldots, d$ i.e. $S_i^{-1}(M) = 0$ for all $i = 1, \ldots, d$. Hence $\exists f_1, \ldots, f_d \in Ann(M)$, where $f_i \in S_i$ and as $f_i . \frac{R}{(f_1, \ldots, f_d)} = 0$, clearly $\frac{R}{(f_1, \ldots, f_d)} \in \mathscr{L}_S$. Hence $ara(\mathscr{L}_S) \leq d$.

If $d = 1, ara(\mathscr{L}_S) \leq 1$. Further if $0 \notin S$, $\mathscr{L}_S$ is a proper subcategory of $R - mod$ and hence by the previous lemma 4.2 $ara(\mathscr{L}_S) \neq 0$. Hence $ara(\mathscr{L}_S) = 1$. □

We state an immediate consequence of prime avoidance and use it to deduce the arithmetic rank of Serre subcategories defined through codimension conditions.

Lemma 4.4. *Let R be a Noetherian ring and I be a proper ideal such that $ht(I) = m \geq 1$. Let $\Gamma \subseteq spec(R)$ be any finite subset such that $\Gamma \cap V(I) = \emptyset$.*
Then $\exists\ b_1, \ldots, b_m$ such that $(b_1, \ldots, b_m) \subseteq I$ where
i) $ht(b_1, \ldots b_i) = i, \forall 1 \leq i \leq m$
and ii) $V(b_1, \ldots, b_m) \cap \Gamma = \emptyset$.

Proof. By hypothesis $ht(I) = m \geq 1$, so that I is not contained in any of the minimal prime ideals of R.Therefore, by prime avoidance, we can find $b_1 \in I \setminus \bigcup_{\Gamma \cup Min(R)} \mathfrak{p}$. In particular, $ht(b_1) = 1$.

For $1 \leq i < m$, assume by induction that $\exists b_1, \ldots, b_i \in I \setminus \bigcup_{\mathfrak{p} \in \Gamma} \mathfrak{p}$ such that $ht(b_1, \ldots, b_j) = j \forall j \leq i$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be minimal primes over $(b_1, \ldots, b_i)$. By Krull's theorem, $ht(\mathfrak{p}_j) \leq i$ for $j = 1, \ldots, r$. Hence $I \nsubseteq \mathfrak{p}_j$, for $j = 1, \ldots, r$, since $ht(I) = m > i$. Hence by prime avoidance, we can choose $b_{i+1} \in I \setminus \bigcup_{j=1}^{r} \mathfrak{p}_j \cup \bigcup_{\mathfrak{p} \in \Gamma} \mathfrak{p}$.

We claim that $ht(b_1, \ldots, b_{i+1}) = i+1$. Let $\mathfrak{q}$ be a minimal prime over $(b_1, \ldots, b_{i+1})$. Since $(b_1, b_2, \ldots, b_i) \subseteq \mathfrak{q}$, we get that $ht(\mathfrak{q}) \geq i$. Note that if $ht(\mathfrak{q}) = i$ then $\mathfrak{q}$ must be a minimal prime of $(b_1, b_2, \ldots, b_i)$ and hence $\mathfrak{q} = \mathfrak{p}_j$. However, this would contradict that $b_{i+1} \notin \mathfrak{p}_j$ and hence $ht(\mathfrak{q}) > i$. Hence, all minimal primes of $(b_1, \ldots, b_{i+1})$ have height at least $i + 1$, thus proving the claim and by induction the lemma. □

Lemma 4.5. *Let R be a Noetherian ring and Γ be any finite subset of $Spec(R)$. Let $\mathscr{L} = \{M \in R - mod | codim(M) \geq m, Supp(M) \cap \Gamma = \emptyset\}$. If $\mathscr{L} \neq \{0\}$, then $ara(\mathscr{L}) = m$.*

Proof. Let $0 \neq M \in \mathscr{L}$. Then $ht(Ann(M)) \geq m$ and $V(Ann(M)) \cap \Gamma = \emptyset$. Hence by lemma 4.4, $\exists b_1, \ldots, b_m$ such that $(b_1, \ldots, b_m) \subseteq Ann(M)$ with $ht(b_1, \ldots, b_m) \geq m$ and $V(b_1, \ldots, b_m) \cap \Gamma = \emptyset$. Hence $\frac{R}{(b_1, \ldots, b_m)} \in \mathscr{L}$. Hence $\mathscr{L}$ satisfies $(**)_m$. Hence $ara(\mathscr{L}) \leq m$.

Note that $ht(a_1, \ldots, a_{m-1}) < m$ for any proper ideal $(a_1, a_2 \ldots, a_{m-1})$ and hence $\frac{R}{(a_1, \ldots, a_{m-1})} \notin \mathscr{L}$. Therefore, $\mathscr{L}$ cannot satisfy $(**)_{m-1}$. Hence $ara(\mathscr{L}) = m$. □

Corollary 4.6. *Let $\mathscr{L}$ be the category of finite length modules over a noetherian equicodimensional (i.e every maximal ideal has the same height) ring R. Then $ara(\mathscr{L}) = dim(R)$.*

Proof. Since R is equicodimensional, the category {finite length modules} is same as the category $\{M \in R - mod | codim(M) \geq dim(R)\}$. Hence by the previous lemma 4.5 , $ara(\mathscr{L}) = dim(R)$. □

Having made these computations of the arithmetic rank, we show its use in the study of our original invariants. To do so, we first recall the main definitions and results of [7] and explain the statements in terms of Serre subcategories and the invariants we have defined. We fix a d-tuple $S = S_1 \times S_2 \times \ldots S_d$ of multiplicatively closed sets for what follows.

Definition 4.2. [7, Defn. 14] Let $P. \in Ch^b(\mathcal{P})$. A $d-$tuple $\alpha = (\alpha^1, \ldots, \alpha^d)$ of families $(\alpha_l^\nu)_{l \in \mathbb{Z}}$ of homomorphisms $\alpha_l^\nu : P_l \to P_{l+1}$ is an S-contraction of $P.$ with weight $s = (s_1, \ldots, s_d) \in S_1 \times \ldots \times S_d$ if $\partial_{l+1}^{P.} \alpha_l^\nu + \alpha_{l-1}^\nu \partial_l^{P.} = s_\nu id_{P_l}$ for all $l \in \mathbb{Z}$ and $\nu = 1, \ldots, d$.

Proposition 4.7. [7, Propn 15] *Every $P. \in Ch^b_{\mathscr{L}_S}(\mathcal{P})$ has an S-contraction (with some weight).*

Using this S-contraction, a Koszul complex with requisite properties is constructed in [7].

Lemma 4.8. [7, Defn 22, Propn 23] *Suppose $P.$ is supported between 0 and n, and has an S-contraction with weight $(s_1, \dots, s_d)$. Then there exists a chain complex homomorphism $\phi : P. \to K.$ where $K. = \Sigma^{n-d} Kos(s_1, \dots, s_d; P_n)$ and ϕ_n is the identity map.*

This allows a reduction of amplitude and hence the main result of [7].

Theorem 4.9. [7, Thm. 42] *Let $\mathscr{L} = \mathscr{L}_S$. Then the maps $\mathcal{I}'_n$ are isomorphisms when $n \geq d$.*

Theorem 4.9 thus proves from our perspective that $\alpha(\mathscr{L}_S) \leq d$. Since our perspective is "dual" to that of [7], we also reinterpret the previous lemma 4.7 and proposition 4.8 for use in what comes next.

Lemma 4.10.
(1) *For $P. \in Ch^{[0,k]}_{\mathscr{L}_S}(\mathcal{P})$ there exists $s_i \in S_i : 1 \leq i \leq d$ and a chain complex morphism $\phi : Kos(s_1, s_2, \dots, s_d) \xrightarrow{\phi} P.$ such that ϕ_0 is an isomorphism.*
(2) *For $n \geq d$, every $P. \in Ch^{[0,n+1]}_{\mathscr{L}_S}(\mathcal{P})$ has a reducer $(Kos(s_1, s_2, \dots, s_d; P_0), \phi)$ where $s_i \in S_i$ and ϕ_0 is an isomorphism.*

Proof. The second statement is a direct consequence of the first. For the first statement, consider $\Sigma^k P.*$ and observe that it lies in $Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$. By proposition 4.7, it has an S-contraction with some weight $s_1, s_2, \dots, s_d$ where $s_i \in S_i$. By lemma 4.8, we can construct a complex $K. = \Sigma^{k-d} Kos(s_1, s_2, \dots, s_d; P_0*)$ with $\phi' : \Sigma^k P.* \to K.$ such that ϕ'_k is identity. Dualizing and shifting, we get $\Sigma^k \phi'^* : \Sigma^k K.* \to P.$ Note that $\Sigma^k K.* = \Sigma^k \Sigma^{d-k} Kos(s_1, s_2, \dots, s_d; P_0*)* = \Sigma^d Kos(s_1, s_2, \dots, s_d; P_0*)*$. Since the Koszul is self-dual, we can choose an isomorphism $Kos(s_1, s_2, \dots, s_d; P_0) \xrightarrow{\sim} \Sigma^d Kos(s_1, s_2, \dots, s_d; P_0*)*$ and we thus obtain an induced map $Kos(s_1, s_2, \dots, s_d; P_0) \xrightarrow{\phi} P.$ such that ϕ_0 is an isomorphism thus proving the lemma. □

Corollary 4.11. $\alpha(\mathscr{L}_S) \leq \beta(\mathscr{L}_S) \leq d$.

We now obtain the promised bound on $\beta(\mathscr{L})$ and hence $\alpha(\mathscr{L})$.

Theorem 4.12. $\alpha(\mathscr{L}) \leq \beta(\mathscr{L}) \leq ara(\mathscr{L})$.

Proof. Note first that if $I = (g_1, g_2, \dots, g_r)$, then letting S_i to be the multiplicatively closed sets consisting of powers of g_i and $S = S_1 \times \dots S_r$, we get that $\mathscr{L}_{V(I)} = \mathscr{L}_S$ and hence by corollary 4.11 $\beta(\mathscr{L}_{V(I)}) \leq r$.

Now let $r \geq ara(\mathscr{L})$. Let $\Lambda = \{V(f_1, \dots, f_r) | \frac{R}{(f_1, \dots, f_r)} \in \mathscr{L}\}$. Then $\mathscr{L}$ satisfies condition $(**)_r$ if and only if $\mathscr{L} = \bigcup_{\lambda \in \Lambda} \mathscr{L}_\lambda$. A direct application of lemma 3.7 along with the previous paragraph now yields the result. □

Corollary 4.13. *Let R be a Noetherian ring and Γ be any finite subset of $Spec(R)$. Let $\mathscr{L} = \{M \in R - mod | codim(M) \geq m, Supp(M) \cap \Gamma = \emptyset\}$. If $\mathscr{L} \neq \{0\}$, then $\alpha_s(\mathscr{L}) = \alpha(\mathscr{L}) = \beta(\mathscr{L}) = m$ and further that $\alpha_i(\mathscr{L}) = 0$.*

Proof. In corollary 4.5 we have seen that $ara(\mathscr{L}) = m$ and it is immmediate that $codim(\mathscr{L}) = m$. Thus, putting together corollary 2.17 and theorem 4.12, we obtain the result. □

Corollary 4.14. *If $\mathscr{L}$ is the Serre subcategory of finite length modules over a noetherian equicodimensional ring R, then $\alpha_s(\mathscr{L}) = \alpha(\mathscr{L}) = \beta(\mathscr{L}) = ara(\mathscr{L}) = dim(R)$ and further that $\alpha_i(\mathscr{L}) = 0$.*

Proof. In corollary 4.6 we have seen that for the Serre subcategory $\mathscr{L}$ of finite length modules over a noetherian equicodimensional ring R , $ara(\mathscr{L}) = dim(R)$. Thus, putting together corollary 2.17 and theorem 4.12, we obtain the result. □

Remark 4.1. Let $x \in R$ and consider $\alpha(\mathscr{L}_{V(x)})$. If x is nilpotent, then we have $V(x) = Spec(R)$ and $\mathscr{L}_{V(x)} = \mathcal{M}$. Hence, $\alpha(\mathscr{L}_{V(x)}) = \beta(\mathscr{L}_{V(x)}) = ara((x)) = 0$.

If x is not nilpotent, then $\mathscr{L}_{V(x)} \neq \mathcal{M}$. By the above lemmas 4.2, 4.12 and 4.1, we know that $0 \neq \alpha(\mathscr{L}_{V(x)}) \leq \beta(\mathscr{L}_{V(x)}) \leq ara((x)) = 1$. Hence, $\alpha(\mathscr{L}_{V(x)}) = \beta(\mathscr{L}_{V(x)}) = 1$.

If x is neither nilpotent nor a unit, then we know further that $\{0\} \neq \mathscr{L}_{V(x)}$ and hence from the above paragraph and lemma 2.17 , we know that $0 \neq \alpha_s(\mathscr{L}_{V(x)}) \leq \alpha(\mathscr{L}_{V(x)}) = \beta(\mathscr{L}_{V(x)}) = 1$ and hence that $\alpha_s(\mathscr{L}_{V(x)}) = \alpha(\mathscr{L}_{V(x)}) = \beta(\mathscr{L}_{V(x)}) = 1$.

The notation $(**)_r$ is derived from condition (*) in [20] which we recall below.

Definition 4.3. [20, Definition 2.11] $\mathscr{L}$ satisfies condition (*) if for every ideal $I \subseteq R$ such that $R/I \in \mathscr{L}$, there exists a regular sequence $f_1, \dots, f_c \subseteq I$ such that $R/(f_1, \dots, f_c) \in \mathscr{L}$.

We end the section with a lemma that describes a relation between these notions and will be used in section 6.

Lemma 4.15. *$\mathscr{L}$ satisfies condition (*) if and only if $grade(\mathscr{L}) = ara(\mathscr{L})$.*

Proof. Suppose $g = grade(\mathscr{L}) = ara(\mathscr{L})$. Then for every $R/I \in \mathscr{L}, \exists f_1, \dots, f_g \in I$ such that $\frac{R}{(f_1,\dots,f_g)} \in \mathscr{L}$. Hence, $\mu(f_1, \dots, f_g) \leq g \leq grade(f_1, \dots, f_g) \leq \mu(f_1, \dots, f_g)$ forcing $g = grade(f_1, \dots, f_g) = \mu(f_1, \dots, f_g)$. But in that case, a standard prime avoidance argument shows that the ideal $(f_1, \dots, f_g)$ is generated by a regular sequence of length g. Hence, $\mathscr{L}$ satisfies condition (*).

Conversely, suppose $\mathscr{L}$ satisfies condition (*) and $g = grade(\mathscr{L})$. Hence, there exists I such that $R/I \in \mathscr{L}$ and $grade(I) = g$. Let $M \in \mathscr{L}$. Hence, by lemma 2.1 $R/Ann(M) \in \mathscr{L}$. Let $J = I \cap Ann(M)$ and observe that $R/J \in \mathscr{L}$. Hence $g \leq grade(J) \leq grade(I) = g$. Hence $grade(J) = g$. Since $\mathscr{L}$ satisfies condition (*), there exists a regular sequence $f_1, \dots, f_c \subseteq J$ such that $R/(f_1, \dots, f_c) \in \mathscr{L}$. But then $g = grade(\mathscr{L}) \leq c \leq grade(J) = g$ and hence $c = g$. Hence, there exists $f_1, \dots, f_g \subseteq J \subseteq Ann(M)$ such that $R/(f_1, \dots, f_g) \in \mathscr{L}$. □

5. Bounds using projective dimension and monomial ideals

In this section, we define an invariant $\beta^{\Delta}_{\underset{\sim}{f}}(I)$ of $\mathscr{L}_{V(I)}$ using reducers based on minimal projective resolutions of powers of a generating set $\underset{\sim}{f}$ for I. This obviously bounds $\beta(\mathscr{L}_{V(I)})$. We further show that when either I is a monomial ideal in a polynomial ring or $char(R) = p > 0$, $\beta^{\Delta}_{\underset{\sim}{f}}(I) = proj\ dim\left(\frac{R}{I}\right)$. This also yields an example of when $\alpha(\mathscr{L}_{V(I)}) = \beta(\mathscr{L}_{V(I)}) = proj\ dim\left(\frac{R}{I}\right) < ara(I)$.

Definition 5.1. (1) Let $\underset{\sim}{f} = \{f_1, \dots, f_d\} \subseteq R$ and define $\underset{\sim}{f}^{[m]} = \{f_1^m, \dots, f_d^m\}$.

(2) Let $I \subseteq R$ be an ideal and $\underset{\sim}{f}$ be a set of generators. Define $I_{\underset{\sim}{f}}^{[m]} := (\underset{\sim}{f}^{[m]})$. Let $\Delta \subseteq \mathbb{N}$ be infinite. Define $\beta_{\underset{\sim}{f}}^{\Delta}(I) = \limsup \{proj\ dim\left(\frac{R}{I^{[m]}}\right) \big| m \in \Delta\}$.

Remark 5.1. (1) The rather cumbersome notation above is used to analyze the already mentioned examples (monomial ideal, char. p) simultaneously.

(2) Note that since projective dimension takes values in a discrete set, $\exists N$ such that $sup\{proj\ dim\left(\frac{R}{I^{[m]}}\right) \big| m \in \Delta, m \geq N\} = \beta_{\underset{\sim}{f}}^{\Delta}(I)$. In particular, it means that $proj\ dim\left(\frac{R}{I^{[m]}}\right) \leq \beta_{\underset{\sim}{f}}^{\Delta}(I) \quad \forall m \geq N, \quad m \in \Delta$.

Example 5.1. Let $R = k[X_1, \dots, X_n]$ and I be a monomial ideal. Let $\underset{\sim}{f}$ be a set of monomials that generates I and Δ be any (infinite) subset. Let $P.$ be the minimal free resolution of R/I. Since the map $\psi : R \to R$ sending $X_i \mapsto X_i^m$ is flat, the minimal free resolution of $\frac{R}{I^{[m]}}$ can be obtained from that of R/I by tensoring $P.$ by R via ψ (see e.g. [16]). Hence, $proj\ dim\left(\frac{R}{I}\right) = proj\ dim\left(\frac{R}{I^{[m]}}\right) \forall m$ and hence $\beta_{\underset{\sim}{f}}^{\Delta}(I) = proj\ dim\left(\frac{R}{I}\right)$.

Example 5.2. When $char(R) = p > 0$, $I^{[p^r]} = I^{p^r}$ and using the the Frobenius map yields $proj\ dim\left(\frac{R}{I}\right) = proj\ dim\left(\frac{R}{I^{[p^r]}}\right)$. Hence for $\Delta = \{p^r | r \in \mathbb{N} \cup \{0\}\}$ we get that $\beta_{\underset{\sim}{f}}^{\Delta}(I) = proj\ dim\left(\frac{R}{I}\right)$ for any generating set $\underset{\sim}{f}$ of I.

5.1. **Relation with projective dimension.**

Theorem 5.1. *Let $\mathscr{L} = \mathscr{L}_{V(I)}$. Suppose for some generating set $\underset{\sim}{f} = (f_1, \dots, f_d)$ of I and every $m' \in \mathbb{N}$, there exists $m \in \mathbb{N}$ and a map $\phi : Q. \to Kos\left(\frac{R}{I^{[m']}}\right)$ where $Q.$ is a projective resolution of $\frac{R}{I^{[m]}}$ with $Q_0 = R$ and $\phi_0 = id$. Then $\beta(\mathscr{L}) \leq \beta_{\underset{\sim}{f}}^{\Delta}(I)$ for every infinite $\Delta \subseteq \mathbb{N} \cup \{0\}$.*

Proof. Let $\Delta \subseteq \mathbb{N}$ be infinite. Note first of all that for any $n \geq m$, there is a natural surjection $\frac{R}{I^{[n]}} \to \frac{R}{I^{[m]}}$ inducing a map on the projective resolutions. Further, if the resolutions have R in degree 0, then it is an isomorphism at the 0^{th} degree. Hence, we can assume further that the m in the hypothesis satisfies that $m \in \Delta$ and $m >> 0$. Note further that for any projective resolution $Q.$, standard arguments with syzygies and Schanuel's lemma tell us that there is a minimal projective resolution which is a summand of $Q.$ and for which all but the last (in particular the 0^{th}) terms are the same. Hence, w.l.o.g. we can assume further that $Q.$ is a minimal projective resolution of $\frac{R}{I^{[m]}}$ with $m >> 0$ and $m \in \Delta$. In particular, this means $Q.$ concentrated in degrees 0 to $proj\ dim\left(\frac{R}{I^{[m]}}\right)$.

Suppose $n \geq \beta_{\underset{\sim}{f}}^{\Delta}(I)$. By Remark 5.1, $\exists N$ such that $\forall m \geq N, m \in \Delta$, we have $proj\ dim\left(\frac{R}{I^{[m]}}\right) \leq \beta_{\underset{\sim}{f}}^{\Delta}(I) \leq n$. Let $P. \in Ch_{\mathscr{L}}^{[0,n+1]}(\mathcal{P})$. Choosing S_i to be the multiplicatively closed set of powers of f_i, lemma 4.10 yields $\psi : Kos\left(\frac{R}{I^{[m']}}\right) \otimes P_0 \to P.$, with ψ_0 an isomorphism. By hypothesis and the previous paragraph, there exists $\phi : Q. \to Kos\left(\frac{R}{I^{[m']}}\right)$ where $Q. \in Ch_{\mathscr{L}}^{[0,n]}(\mathcal{P})$ with $\phi_0 = id$. Composing with ϕ, we get a reducer $(Q., \psi \circ \phi)$ for $P.$

The proof is now complete by using Theorem 3.5. ∎

Corollary 5.2. *Suppose $char(R) = p > 0$. Let $\mathscr{L} = \mathscr{L}_{V(I)}$ and suppose for every $m' \in \mathbb{N}$, there exists $m \in \mathbb{N}$ and a map $\phi : Q. \to Kos\left(\frac{R}{I^{[m']}}\right)$ where $Q.$ is a projective resolution of $\frac{R}{I^{[m]}}$ with $Q_0 = R$ and $\phi_0 = id$. Then $\alpha(\mathscr{L}) \leq \beta(\mathscr{L}) \leq proj\ dim\left(\frac{R}{I}\right)$.*

Proof. The first inequality is theorem 3.6 and the second is an immediate consequence of theorem 5.1 by choosing any generating set of I and $\Delta = \{p^r | r \in \mathbb{N} \cup \{0\}\}$ as in Example 5.2. ∎

5.2. **Improved bounds for monomial ideals.** Let $R = k[X_1, X_2, \ldots, X_n]$ and $\underset{\sim}{f} = \{f_1, f_2, \ldots, f_d\}$ be monomials in R. Let $K.$ be the corresponding Koszul complex. Note that the l^{th} term K_l is the free module with basis $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}\}$ where $i_j \in \mathbb{Z}$ and $1 \leq i_1 \leq i_2 \leq \ldots i_l \leq d$. The differential d_l^K takes the basis vector $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}$ to $\sum_{j=1}^{l} (-1)^{j-1} f_{i_j} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_l}$.

We consider the Taylor resolution $T.$ corresponding to the above set of monomials as defined in [12]. Note that the l^{th} term T_l is the free module with basis $\{e_{i_1} e_{i_2} \ldots e_{i_l}\}$ where $i_j \in \mathbb{Z}$ and $1 \leq i_1 \leq i_2 \leq \ldots i_l \leq d$. The differential d_l^T is defined by :

$$d_l^T(e_{i_1} e_{i_2} \ldots e_{i_l}) = \sum_{j=1}^{l} (-1)^{j-1} \frac{lcm(f_{i_1}, \ldots, f_{i_{j-1}}, f_{i_j}, f_{i_{j+1}}, \ldots, f_{i_l})}{lcm(f_{i_1}, \ldots, f_{i_{j-1}}, \widehat{f_{i_j}}, f_{i_{j+1}}, \ldots, f_{i_l})} e_{i_1} \ldots \widehat{e_{i_j}} \ldots e_{i_l}.$$

For $m \in \mathbb{N}$, let $T.^{[m]}$ be the Taylor resolution of $\underset{\sim}{f}^{[m]}$. For $m >> 0$, We will exhibit a morphism from $T.^{[m]}$ to $K.$ To this end, we prove the next lemma.

Lemma 5.3. *Let $f_1 f_2 \ldots f_d = \prod_{i=1}^{n} X_i^{m_i}$. Let $m \geq \max\{m_i\}$. Then for every l-tuple , $1 \leq i_1 < \ldots < i_l \leq d$, $f_{i_1} f_{i_2} \ldots f_{i_l}$ divides $lcm(f_{i_1}^m, f_{i_2}^m, \ldots, f_{i_l}^m)$.*

Proof. Suppose $lcm(f_{i_1}^m, f_{i_2}^m, \ldots, f_{i_l}^m) = \prod_{i=1}^{n} X_i^{k_i}$ and $f_{i_1} f_{i_2} \ldots f_{i_l} = \prod_{i=1}^{n} X_i^{l_i}$.

If $k_i > 0$, then X_i occurs in some f_{i_j} and hence $k_i \geq m \geq m_i \geq l_i$. If $k_i = 0$, then X_i does not occur in any f_{i_j} and hence also not in the expression $f_{i_1} f_{i_2} \ldots f_{i_l}$ and hence $l_i = 0$. Hence, $k_i \geq l_i \quad \forall i$ which proves the lemma. ∎

Theorem 5.4. *Let $R = k[X_1, \ldots, X_n]$ and $I = (f_1, \ldots, f_d)$ be a monomial ideal. Then with the above notations, for $m >> 0$, there exists $\phi : T.^{[m]} \to K.$, where ϕ_0 is identity.*

Proof. Define $\phi_0 : T_0^{[m]} \to K_0$ to be the identity and for all $1 \leq l \leq d$, define ϕ_l on the generators by :

$$\phi_l(e_{i_1} e_{i_2} \ldots e_{i_l}) = \frac{lcm(f_{i_1}^m, f_{i_2}^m, \ldots, f_{i_l}^m)}{f_{i_1} f_{i_2} \ldots f_{i_l}} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}$$

for all $1 \leq i_1 < \ldots < i_l \leq d$. Note that for $m >> 0$ by lemma 5.3, the coefficient is in R. We now check that the squares commute.

$$\begin{aligned}
&\phi_{l-1} d_l^{T^{[m]}}(e_{i_1} e_{i_2} \ldots e_{i_j} \ldots e_{i_l}) \\
&\quad = \phi_{l-1}\left(\sum_{j=1}^{l}(-1)^{j-1}\frac{lcm(f_{i_1}^m,\ldots,f_{i_{j-1}}^m,f_{i_j}^m,f_{i_{j+1}}^m,\ldots,f_{i_l}^m)}{lcm(f_{i_1}^m,\ldots,f_{i_{j-1}}^m,\widehat{f_{i_j}}^m,f_{i_{j+1}}^m,\ldots,f_{i_l}^m)} e_{i_1}\ldots\widehat{e_{i_j}}\ldots e_{i_l}\right) \\
&\quad = \sum_{j=1}^{l}(-1)^{j-1}\frac{lcm(f_{i_1}^m,\ldots,f_{i_{j-1}}^m,f_{i_j}^m,f_{i_{j+1}}^m,\ldots,f_{i_l}^m)}{lcm(f_{i_1}^m,\ldots,f_{i_{j-1}}^m,\widehat{f_{i_j}^m},f_{i_{j+1}}^m,\ldots f_{i_l}^m)}\times \\
&\qquad\qquad \frac{lcm(f_{i_1}^m,\ldots,f_{i_{j-1}}^m,\widehat{f_{i_j}^m},f_{i_{j+1}}^m,\ldots,f_{i_l}^m)}{f_{i_1}\cdots f_{i_{j-1}}\widehat{f_{i_j}}f_{i_{j+1}}\cdots f_{i_l}} e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge\widehat{e_{i_j}}\ldots\wedge e_{i_l} \\
&\quad = \sum_{j=1}^{l}(-1)^{j-1}\frac{lcm(f_{i_1}^m,\ldots,f_{i_{j-1}}^m,f_{i_j}^m,f_{i_{j+1}}^m,\ldots,f_{i_l}^m)}{f_{i_1}\cdots f_{i_{j-1}}\widehat{f_{i_j}}f_{i_{j+1}}\cdots f_{i_l}} e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge\widehat{e_{i_j}}\ldots\wedge e_{i_l}
\end{aligned}$$

$$\begin{aligned}
&d_l^K\phi_l(e_{i_1}e_{i_2}\ldots e_{i_l}) = d_l^K\left(\frac{lcm(f_{i_1}^m,f_{i_2}^m,\ldots,f_{i_l}^m)}{f_{i_1}f_{i_2}\cdots f_{i_l}} e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge e_{i_l}\right) \\
&\quad = \frac{lcm(f_{i_1}^m,f_{i_2}^m,\ldots,f_{i_l}^m)}{f_{i_1}f_{i_2}\cdots f_{i_l}}\sum_{j=1}^{l}(-1)^{j-1}f_{i_j}e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge\widehat{e_{i_j}}\ldots\wedge e_{i_l} \\
&\quad = \sum_{j=1}^{l}(-1)^{j-1}\frac{lcm(f_{i_1}^m,\ldots,f_{i_{j-1}}^m,f_{i_j}^m,f_{i_{j+1}}^m,\ldots,f_{i_l}^m)}{f_{i_1}\cdots f_{i_{j-1}}\widehat{f_{i_j}}f_{i_{j+1}}\cdots f_{i_l}} e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge\widehat{e_{i_j}}\ldots\wedge e_{i_l}
\end{aligned}$$

Hence, the squares commute. This proves the theorem. □

Theorem 5.5. *Let $R = k[X_1, \ldots, X_n]$ where k is a field, $I \subseteq R$ be a monomial ideal and $\mathscr{L} = \mathscr{L}_{V(I)}$. Then $\alpha(\mathscr{L}) \leq \beta(\mathscr{L}) \leq proj\ dim\left(\frac{R}{I}\right)$.*

Proof. Choose a monomial generating set $\underset{\sim}{f}$ of I. Theorem 5.4 shows that $\mathscr{L}$ satisfies the hypothesis of theorem 5.1. Hence, $\alpha(\mathscr{L}) \leq \beta(\mathscr{L}) \leq \beta^{\Delta}_{\underset{\sim}{f}}(I)$ for any infinite $\Delta \subseteq \mathbb{N}$. However, as noted in example 5.1 , $\beta^{\Delta}_{\underset{\sim}{f}}(I) = proj\ dim\left(\frac{R}{I}\right)$ and hence $\alpha(\mathscr{L}) \leq \beta(\mathscr{L}) \leq proj\ dim\left(\frac{R}{I}\right)$ thus completing the proof. □

Note that by [13] if I is a squarefree monomial ideal, then $proj\ dim\left(\frac{R}{I}\right) \leq ara(I)$ and hence this is an enhanced bound.

As promised in the beginning of the section, we use this bound to provide an example from [25] showing that neither α nor β are the arithmetic rank.

Example 5.3. Let $R = k[X_1, X_2, X_3, X_4, X_5, X_6]$ and I be the squarefree monomial ideal of R generated by the following elements: $X_1X_2X_3, X_1X_2X_5, X_1X_3X_6, X_1X_4X_5, X_1X_4X_6, X_2X_3X_4, X_2X_4X_6, X_2X_5X_6, X_3X_4X_5, X_3X_5X_6$.

Let $\mathscr{L} = \mathscr{L}_{V(I)}$. It is shown in [25] that

$$proj\ dim\left(\tfrac{R}{I}\right) = \begin{cases} 3 & \text{if} \quad char(k) \neq 2 \\ 4 & \text{if} \quad char(k) = 2 \end{cases} \qquad \text{and } ara(I) = 4.$$

Hence $\alpha(\mathscr{L}) \leq \beta(\mathscr{L}) \leq 3$ if $char(k) \neq 2$. In fact $codim(\mathscr{L}) = ht(I) = 3$ and since $K_0(R \text{ on } \mathscr{L}) \neq 0$, using lemma 2.17 we get that $\alpha_s(\mathscr{L}) = \alpha(\mathscr{L}) = \beta(\mathscr{L}) = 3$.

Hence if $char(k) \neq 2, \alpha_s(\mathscr{L}) = \alpha(\mathscr{L}) = \beta(\mathscr{L}) = 3 < 4 = ara(I)$.

6. Answers to Question 2 and remarks about the invariants

6.1. **Remarks on the invariants.** Example 5.3 shows that $\beta(\mathscr{L}_{V(I)})$ and hence $\alpha(\mathscr{L}_{V(I)})$ and hence $\alpha_s(\mathscr{L}_{V(I)})$ can be strictly less than the arithmetic rank. Further, by remark 4.1 , $\alpha_s(\mathscr{L}_{V(x)}) = 1 > 0 = codim(\mathscr{L}_{V(x)})$ when x is a zero divisor which is not nilpotent. This shows that $\alpha_s(\mathscr{L}_{V(I)})$ and hence α and hence β can be strictly larger than the codimension. Thus, α_s, α and β all appear interesting new invariants (even for ideals) and we are as yet unaware of a situation where they differ.

Note that by lemma 2.4 , we know that for $n < grade(\mathscr{L})$, We get that $\dfrac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} = 0$ and hence $\mathcal{I}'_n : \dfrac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \to \dfrac{K_0\left(Ch^{[0,n+1]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$ is injective. Thus, $\alpha_i(\mathscr{L})$ is either between $grade(\mathscr{L})$ and $\alpha(\mathscr{L})$ or is 0. In particular, we get the following lemma :

Lemma 6.1. *If $grade(\mathscr{L}) = \alpha(\mathscr{L})$, then $\alpha_i(\mathscr{L}) = 0$.*

Indeed, all the examples in which we can compute α_i (e.g. corollary 4.13), we see that $\alpha_i(\mathscr{L}) = 0$ since the above hypothesis is satisfied. As a resullt, in all those examples, $\alpha_s(\mathscr{L})$ equalled $\alpha(\mathscr{L})$.

6.2. **Answers to question 2.** In this subsection, we answer question 2. For simplicity of notation, let g be $grade(\mathscr{L})$. Recall that $Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})$ is the full subcategory of $Ch^{b}_{\mathscr{L}}(\mathcal{P})$ consisting of complexes $P.$ with $H_i(P.) = 0, i \neq 0$. Note that $Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})$ is an exact category closed under kernels of surjections. Further, by lemma 2.4 , we have a natural inclusion of full subcategories $Ch^{[0,g]}_{\mathscr{L}}(\mathcal{P}) \subseteq Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P}) \subseteq Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})$, inducing group homomorphisms

$$\frac{K_0\left(Ch^{[0,g]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \xrightarrow{u} \frac{K_0\left(Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \xrightarrow{v} \frac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \quad \text{and} \quad v \circ u = \nu_g$$

where ν_g is as in corollary 2.15. Recall that $\mathcal{H} \cap \mathscr{L}$ is the full subcategory of modules in $\mathscr{L}$ with finite projective dimension and H_0 induces an exact functor from $Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P}) \to \mathcal{H} \cap \mathscr{L}$ and hence a homomorphism $K_0(Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})) \to K_0(\mathcal{H} \cap \mathscr{L})$. Clearly this map is surjective and the subgroup of exact complexes is in the kernel, yielding a surjective homomorphism $\dfrac{K_0\left(Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle} \xrightarrow{H_0} K_0(\mathcal{H} \cap \mathscr{L})$.

Lemma 6.2. *v factors through H_0.*

Proof. Any element ω of $K_0(Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P}))$ is of the form $\omega = [P.] - [P.']$ where $P., P.' \in Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})$. Suppose $H_0(\omega) = 0$ in $K_0(\mathcal{H} \cap \mathscr{L})$. Then $P., P.'$ are resolutions of the same module in $\mathscr{L}$ and hence by lifting the identity map, we get a quasi-isomorphism $P. \xrightarrow{f} P.'$. Then by lemma 2.11 , we have that $[P.] = [P.']$ in $\frac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] | P. \text{ is exact } \rangle}$, i.e. $v(\omega) = 0$. Hence v factors through H_0. ∎

Let us denote the induced map by $\mathcal{R}$. Let $n \geq \alpha(\mathscr{L})$. Using the notations from corollary 2.15 and letting $w_n = \mathcal{I}'_{n-1} \circ \ldots \circ \mathcal{I}'_g$, we thus get the following commutative diagram of groups :

$$\begin{array}{ccccccc}
 & & \frac{K_0\left(Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] \mid P. \text{ is exact } \rangle} & \xrightarrow{H_0} & K_0(\mathcal{H}\cap\mathscr{L}) & & \\
 & \nearrow^{u} & & & \downarrow{\mathcal{R}} & & \\
\frac{K_0\left(Ch^{[0,g]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] \mid P. \text{ is exact } \rangle} & & \xrightarrow{\nu_g} & & \frac{K_0\left(Ch^{[0,\infty)}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] \mid P. \text{ is exact } \rangle} & \xrightarrow[\sim]{\mathcal{I}} & K_0(R \text{ on } \mathscr{L}) \\
 & \searrow_{w_n} & & & \wr\uparrow{\nu_n} & \nearrow_{\mathcal{I}_n}^{\sim} & \\
 & & & & \frac{K_0\left(Ch^{[0,n]}_{\mathscr{L}}(\mathcal{P})\right)}{\langle [P.] \mid P. \text{ is exact } \rangle} & &
\end{array}$$

Qn. 2 thus asks when $\mathcal{R}$ is an isomorphism. The following theorem gives sufficient conditions in terms of reducers.

Theorem 6.3. *Assume $g = \beta(\mathscr{L})$. Then $\mathcal{R}$ is an isomorphism.*

Proof. If $\mathscr{L} = \{0\}$, both groups are 0. So we can assume that $\mathscr{L} \neq \{0\}$.

If $g = 0$, then $\beta(\mathscr{L}) = 0$ and hence $\mathscr{L} = \mathcal{M}(R)$. In that case, both sides are isomorphic to $K_0(R)$ and $\mathcal{R}$ is clearly an isomorphism.

Thus, we can assume that $\mathscr{L} \neq \{0\}$ and $g > 0$. By corollaries 2.17 and 3.6, $g \leq \alpha(\mathscr{L}) \leq \beta(\mathscr{L})$ and so the hypothesis implies $g = \alpha(\mathscr{L})$ and hence that ν_g is an isomorphism. Since $\nu_g = \mathcal{R} \circ H_0 \circ u$, to complete the proof, it is enough to show that $H_0 \circ u$ is surjective.

Let $M \in \mathcal{H} \cap \mathscr{L}$ and consider its class $[M] \in K_0(\mathcal{H} \cap \mathscr{L})$. If $M = 0$, then $[M] \in image(H_0 \circ u)$. So assume $M \neq 0$ and choose a projective resolution $P.$ of M of amplitude $pd(M)$. Then by corollary 2.8[2], $pd(M) \geq g$. We will show by induction on $pd(M)$ that $[M] \in image(H_0 \circ u)$. If $pd(M) = g$, then this is obvious giving us the base case. Let $pd(M) > g$ and assume that whenever $N \in \mathcal{H} \cap \mathscr{L}$ with $pd(N) < pd(M)$, then $[N] \in image(H_0 \circ u)$. By our hypothesis, there is a reducer $(Q., u)$ for $P.$ such that $Q. \in Ch^{[0,\beta(\mathscr{L})]}_{\mathscr{L}}(\mathcal{P})$. Since $\beta(\mathscr{L}) = g$, corollary 2.8[3] implies that $Q. \in Ch^{[0,\beta(\mathscr{L})]}_{\mathscr{L}}(\mathcal{P}) \cap Ch^{h[0,0]}_{\mathscr{L}}(\mathcal{P})$ and hence $H_0(Q.) \in \mathcal{H} \cap \mathscr{L}$. Since the induced map $H_0(Q.) \xrightarrow{H_0(u)} H_0(P.)$ is surjective, $ker(H_0(u)) \in \mathcal{H} \cap \mathscr{L}$. Further note that since $pd(H_0(Q.)) \leq amplitude(Q.) = \beta(\mathscr{L}) \leq pd(M) - 1$, it follows that $pd(ker(H_0(u))) \leq pd(M) - 1$ and hence $[H_0(Q.)], [ker(H_0(u))] \in image(H_0 \circ u)$. Since $[M] = [H_0(Q.)] - [ker(H_0(u))]$, this shows that $[M] \in image(H_0 \circ u)$ thus completing the proof. □

We draw immediate conclusions from this theorem.

Theorem 6.4. *Suppose $g = ara(\mathscr{L})$. Then $K_0(\mathcal{H} \cap \mathscr{L}) \xrightarrow[\sim]{\mathcal{I} \circ \mathcal{R}} K_0(R \text{ on } \mathscr{L})$.*

Proof. By theorem 4.12 and lemma 2.17 , $g \leq \beta(\mathscr{L}) \leq ara(\mathscr{L})$. The hypothesis thus implies $g = \beta(\mathscr{L})$ and hence by theorem 6.3 $\mathcal{R}$ is an isomorphism. □

Indeed all the situations considered in [7, Table, Pg. 4] (even without the local hypothesis) and several others (e.g. corollaries 4.13 , 4.14) satisfy the hypothesis of Theorem 6.4 . In fact, by lemma 4.15 , $\mathscr{L}$ satisfies condition (*) and hence by [20, Theorem 1.2], there is an equivalence of the underlying derived categories (or rather their model structures) $D^b(\mathcal{H} \cap \mathscr{L})$ and $D^b_{\mathscr{L}}(\mathcal{P})$ which induces the equivalence $\mathcal{R}$.

One may wonder if the sufficient condition $g = ara(\mathscr{L})$ is also necessary. Theorem 6.3 applied to example 5.3 shows that is not the case.

Lemma 6.5. *Let $R = k[X_1, X_2, X_3, X_4, X_5, X_6]$ and I be the squarefree monomial ideal of R generated by the following elements: $X_1X_2X_3, X_1X_2X_5, X_1X_3X_6,$ $X_1X_4X_5, X_1X_4X_6, X_2X_3X_4, X_2X_4X_6, X_2X_5X_6, X_3X_4X_5, X_3X_5X_6$. Then $K_0(\mathcal{H} \cap \mathscr{L}) \xrightarrow[\sim]{\mathcal{I} \circ \mathcal{R}} K_0(R \text{ on } \mathscr{L})$.*

Since R is regular, this is of course well-known, but we believe the technique of reducers will yield similar results in a wide range of situations and this example lends credence to the belief.

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