TATE RESOLUTIONS AND DERIVED EQUIVALENCES

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Abstract. For any finite sequence of elements $\underset{\sim}{s}$ in a commutative Noetherian ring and a finitely generated module M, we define the Tate resolution $T(\underset{\sim}{s};M)$. We show that for $n \gg 0$, the natural map from the Koszul complex $K(\underset{\sim}{s}^n;M)$ to the Koszul complex $K(\underset{\sim}{s};M)$ factors through $T(\underset{\sim}{s}^n;M)$. We define ideals having eventually finite projective dimension (efpd), which includes ideals in regular rings, set-theoretic complete intersections and ideals with finite projective dimension in prime characteristic. We show a certain derived equivalence, which, amongst other results, yields an equivalence of the bounded derived category of finite projective dimension modules supported on an ideal having efpd with the bounded derived category of projective modules with homologies supported on the same ideal.

1. Introduction

Let R denote a commutative Noetherian ring and M be a finitely generated R-module. Let $\underset{\sim}{s} = s_1, \ldots, s_d \in R$ be any sequence of elements and let $K(\underset{\sim}{s};M)$ be the Koszul complex on $\underset{\sim}{s}$ with coefficients in M . We denote $K(\underset{\sim}{s};R)$ by $K(\underset{\sim}{s})$. Recall that the Tate resolution $T(\underset{\sim}{s})$ is obtained by resolving successive homologies starting from $K(\underset{\sim}{s})$ to form a projective resolution of $R/(\underset{\sim}{s})$. In fact, both complexes are differential graded algebras, and by construction, there is a natural inclusion $K(\underset{\sim}{s}) \to T(\underset{\sim}{s})$ which is even a dg-algebra morphism. Similar to the Tate resolution, we define the chain complex $T(\underset{\sim}{s};M)$ in Definition-Remark 3.4. $T(\underset{\sim}{s};M)$ is a (not necessarily projective) resolution of $M/(\underset{\sim}{s})M$ and contains $K(\underset{\sim}{s};M)$.

It is well-known that $K(\underset{\sim}{s})$ and $T(\underset{\sim}{s})$ "coincide" if and only if $\underset{\sim}{s}$ is a regular sequence. In other words, an inverse $T(\underset{\sim}{s}) \dashrightarrow K(\underset{\sim}{s})$ exists if and only if $\underset{\sim}{s}$ is a regular sequence. Thus when $\underset{\sim}{s}$ is not a regular sequence, such a map cannot exist. Now let $\underset{\sim}{s}^n$ denote the sequence obtained from $\underset{\sim}{s}$ by taking n^{th} powers of each of the elements of $\underset{\sim}{s}$. Note that the natural surjection $R/(\underset{\sim}{s}^n) \twoheadrightarrow R/(\underset{\sim}{s})$ induces a natural chain complex map $\eta : K(\underset{\sim}{s}^n;M) \to K(\underset{\sim}{s};M)$ which is the identity map in the case $n = 1$. Our main theorem Theorem 3.5 shows that for *any* sequence $\underset{\sim}{s}$, for sufficiently large n, this map factors through $T(\underset{\sim}{s}^n;M)$ i.e.

$$\begin{tikzcd}
K(\underset{\sim}{s}^n;M) \arrow[r, hook] \arrow[dr, "\eta"'] & T(\underset{\sim}{s}^n;M) \arrow[d, dashed, "\phi"] . \\
& K(\underset{\sim}{s};M)
\end{tikzcd}$$

When $M = R$ and $\underset{\sim}{s} = \{s\}$, a singleton set, this phenomenon and the role of raising powers and R being Noetherian can be seen as follows : Since R is Noetherian, there exists $l \in \mathbb{N}$

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such that $Ann(s^l) = Ann(s^{l+1})$. The diagram below shows the map ϕ when $n = l+1$.

$$\begin{tikzcd}
T(s^{l+1}): \arrow[r] & R^t \arrow[r, "\partial"] \arrow[d, "0"] & Re \arrow[r, "(s^{l+1})"] \arrow[d, "(s^l)"] & R \arrow[r] \arrow[d, "id"] & 0 . \\
K(s): \arrow[r] & 0 \arrow[r] & Re \arrow[r, "(s)"] & R \arrow[r] & 0
\end{tikzcd}$$

Since ∂ maps the basis vectors of R^t to the generators of $Ann(s^{l+1}) = Ann(s^l)$, we can see that ϕ is a chain complex map.

The proof of the main theorem in section 3 relies on a vanishing theorem for the map of Tor modules from powers of an ideal to the ideal, i.e. the map $\mathrm{Tor}_i^R(R/I^n, M) \to \mathrm{Tor}_i^R(R/I^r, M)$ is zero for $n \gg 0$. This is used to prove Lemma 3.2. Lemma 3.2 and Theorem 3.3 are statements about when certain divisibility properties of elements in the syzygy of a Koszul complex on $\underset{\sim}{s}$ ensures that it is in in the image of the natural map from a Koszul complex on $\underset{\sim}{s}^n$ for some large power n. The essence of why this works is captured in 2.2(ii), which shows that any element r acts like a non-zero divisor on the submodule r^nM for $n \gg 0$. Lemma 3.2 and Theorem 3.3 are the technical heart of what makes the main theorem work.

It is well-known that vanishing theorem for the map of Tor modules implies a similar theorem about the vanishing of the maps between Koszul homologies. Such vanishing statements for the maps between Tor modules have been studied before e.g. in [And74], [EH05] and [AHS15]. Such theorems rely on the Artin-Rees lemma (refer subsection 2.1 for more details) and the strength of the statement depends on the best possible choice of n as a function of r. As a consequence of our main theorem, we show that the vanishing of the maps between Koszul homologies also implies the vanishing of the maps between Tor modules as above. In particular, we obtain a function $w(r)$ (refer Theorem 3.10 and its proof) such that $\mathrm{Tor}_i^R(R/I^{w(r)}, M) \to \mathrm{Tor}_i^R(R/I^r, M)$ is zero for all $i \geq 1$. We note that such results are known for local rings with a linear function (refer [EH05], [AHS15]) but this appears to be new for general (i.e. not necessarily local) rings. As a consequence, we show that for a principal ideal I in any Noetherian ring R, every R-module M is syzygetically Artin-Rees w.r.t. I.

Our interest in obtaining the main theorem was its potential use in generalizing some known equivalences of derived categories. While studying this question (which is described in the next paragraph), we were naturally led to an intriguing collection of ideals which we have defined as ideals of eventually finite projective dimension (refer Defn 2.14). This class includes set-theoretic complete intersections as well as ideals of finite projective dimension in prime characteristic $p > 0$ and ideals in a regular ring. We refer to subsection 2.3 for more details and examples of such ideals.

We now recall some notations in order to discuss the afore-mentioned derived equivalences. Let $mod(R)$ denote the category of finitely generated R-modules. Let $\mathcal{P}$ denote the full subcategory of projective modules and $\mathcal{A}$ denote a resolving subcategory of $mod(R)$. For a subcategory $\mathcal{K} \subset mod(R)$, let $Ch^b(\mathcal{K})$ denote the category of bounded chain complexes in $\mathcal{K}$. For a Serre subcategory $\mathscr{L}$, let $Ch^b_{\mathscr{L}}(\mathcal{K})$ be the full subcategory of $Ch^b(\mathcal{K})$ consisting of complexes with homologies in $\mathscr{L}$ and $D^b(\mathcal{K})$ denote the corresponding derived category obtained by inverting quasi-isomorphisms. We have the natural functor $D^b(\overline{\mathcal{A}} \cap \mathscr{L}) \rightsquigarrow D^b_{\mathscr{L}}(\mathcal{A})$, which sends a complex to its projective resolution. In [SS17], it was proved that for certain Serre subcategories (e.g. when $\mathscr{L} = \mathscr{L}_{V(I)}$ where I is a set-theoretic complete intersection), the above functor is an equivalence. As a consequence of our main theorem above, we can generalize the above-mentioned result in [SS17] to a much larger class of Serre subcategories (refer Thm 4.5). In particular, we prove the following theorem (Theorem 4.7):

Theorem 1.1. *Let R be a commutative Noetherian ring. Let $\mathcal{A} \subseteq mod(R)$ be a resolving subcategory. Then there is an equivalence of categories $D^b(\overline{\mathcal{A}} \cap \mathscr{L}) \rightsquigarrow D^b_{\mathscr{L}}(\mathcal{A})$ in the following cases :*

(1) R is a regular ring.
(2) $\mathscr{L}$ satisfies condition $(*)$ defined in [SS17, Definition 2.11].
(3) $\mathscr{L} = \mathscr{L}_{V(K)}$ where K has efpd.

In particular, $D^b(\overline{\mathcal{P}} \cap \mathscr{L}) \simeq D^b_{\mathscr{L}}(\mathcal{P})$ in these cases.

Theorem 1.1 consolidates both the known cases of the derived equivalence, namely when the ring is regular and [SS17, Theorem 4.5], and extends it to ideals having efpd.

We pause to mention the connection between the Artin-Rees lemma and the derived equivalence in a classical case. In [Gro77], there are conditions under which $D^b(\mathcal{B}) \to D^b_{\mathcal{B}}(\mathcal{A})$ is an equivalence, where $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is a Serre subcategory. In [Kel99, Pg. 17, Example (b)], it is checked that the conditions hold when $\mathcal{A} = mod(R)$ and $\mathcal{B}$ is the full subcategory of I-torsion modules. The proof crucially uses the Artin-Rees lemma.

We now describe the consequences of the main theorem for certain K_0 groups. The functor above induces a homomorphism $K_0\left(D^b(\overline{\mathcal{P}} \cap \mathscr{L})\right) \to K_0\left(D^b_{\mathscr{L}}(\mathcal{P})\right)$. In [CS22], it was shown that the right side is a direct limit of maps

$$\frac{K_0(Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P}))}{\langle P_\bullet \mid P_\bullet \text{ is exact}\rangle} \xrightarrow{I'_k} \frac{K_0(Ch^{[0,k+1]}_{\mathscr{L}}(\mathcal{P}))}{\langle P_\bullet \mid P_\bullet \text{ is exact}\rangle}$$

and a derived version of the stable range $\alpha(\mathscr{L})$ was introduced. Further, a related invariant $\beta(\mathscr{L})$ based on a notion called a reducer was defined and studied in order to give bounds on the derived stable range $\alpha(\mathscr{L})$. (refer Section 2) for more details). Using Theorem 3.5, it can be shown that $\beta(\mathscr{L})$ and hence $\alpha(\mathscr{L})$ can be bounded by the projective dimension of R/I when $\mathscr{L} = \mathscr{L}_{V(I)}$ and I is a perfect ideal.

In [SS17], the derived equivalence above was also used to characterize Cohen-Macaulay local rings. We add to this characterization in corollary 5.3, which is obtained as a consequence of the series of implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ for an ideal $I \neq R$ and $\mathscr{L} = \mathscr{L}_{V(I)}$ where the implications are :

(i) I is an ideal of eventually finite projective dimension.
(ii) There is a derived equivalence $D^b(\overline{\mathcal{P}} \cap \mathscr{L}) \simeq D^b_{\mathscr{L}}(\mathcal{P})$.
(iii) There exists a non-zero finitely generated module M with finite projective dimension and $Supp(M) = V(I)$.
(iv) For every minimal prime $\mathfrak{p}$ of I, $R_{\mathfrak{p}}$ is Cohen-Macaulay.

This appears as lemma 5.1. This also leads to the question of whether the reverse implications are true.

We briefly discuss the outline of the article. In section 2, we first mention basic facts from commutative algebra and statements related to the Artin-Rees property and the vanishing of maps between Tor modules. We then state the definition of ideals having efpd and discuss some properties and examples. We then state preliminaries related to categories, K_0, etc. In section 3, we prove the main theorem after preparatory lemmas related to vanishing of maps between Tor modules and similarly lemmas related to vanishing of maps between Koszul modules. In section 4, we prove the derived equivalence mentioned earlier and in particular, prove theorem 1.1 mentioned above. In section 5, we prove the implications mentioned above and add to the characterization of Cohen-Macaulay local rings. Finally, we also briefly discuss known results and counter-examples related to ideals having efpd.

2. Preliminaries

Let us fix some notations: Let $d, n \in \mathbb{N}$ and $\mathcal{P}_m^d$ be the set of all ordered m-subsets of $[d] = \{1, \ldots, d\}$. Let $s_1, \ldots, s_d \in R$. Denote $\underset{\sim}{s}^n = \{s_1^n, \ldots, s_d^n\}$ and $\underset{\sim}{s} = \underset{\sim}{s}^1$. Recall that the Koszul complex $K(\underset{\sim}{s}^n) = \bigwedge(\oplus_{i=1}^d Re_i)$ is a differential graded algebra with $\deg(e_i) = 1$ for all i and the differential mapping e_i to s_i^n. Hence the algebra structure on $K(\underset{\sim}{s}^n)$ is the exterior algebra for all $n \in \mathbb{N}$. We denote the Koszul complex on $\underset{\sim}{s}^n$ with coefficients in M by

$$K(\underset{\sim}{s}^n; M) = \left(K_m(\underset{\sim}{s}^n; M), \partial_m^{K(\underset{\sim}{s}^n;M)}\right)_{m=0}^d.$$

A typical element of $K_m(\underset{\sim}{s}^n; M) = \bigoplus_{\underset{\sim}{i} \in \mathcal{P}_m^d} Me_{\underset{\sim}{i}}$ is of the form $\sum_{\underset{\sim}{i} \in \mathcal{P}_m^d} c_{\underset{\sim}{i}} e_{\underset{\sim}{i}}$ where $c_{\underset{\sim}{i}} \in M$. Some authors denote $H_m(K(\underset{\sim}{s}^n; M))$ by $H_m(\underset{\sim}{s}^n; M)$.

In the introduction, a basic example of how chains of annihilator ideals are useful was demonstrated. In a similar spirit, the modules $(0 :_M s^t)$ will play an important role in the article. We state some basic properties of these submodules.

Notation 2.1. Since R is Noetherian and M is a finitely generated R-module, for each $s \in R$, the ascending chain of submodules of M,

$$(0 :_M s) \subseteq \cdots \subseteq (0 :_M s^t) \subseteq (0 :_M s^{t+1}) \subseteq \cdots$$

stabilizes. Let $l_i \in \mathbb{N}$ be such that $(0 :_M s_i^{l_i}) = (0 :_M s_i^{l_i+k})$ for all $k \in \mathbb{N}$. We define

$$l = l(\underset{\sim}{s}) := max\{l_i \mid i \in [d]\}.$$

Lemma 2.2. *(i) Let $k \geq 1$. Then* $\left(0 :_M \left(\prod_{t \in [d]} s_t^{l+k}\right)\right) = \left(0 :_M \left(\prod_{t \in [d]} s_t^{l}\right)\right)$.

(ii) Let $x, y \in M$ such that s_i^l divides x and y. If $s_i x = s_i y$, then $x = y$.

Proof. (i) Let $m \in M$ such that $\left(\prod_{t \in [d]} s_t^{l+k}\right) m = 0$. We have $(0 :_M s_i^{l_i}) = (0 :_M s_i^{l_i+k})$ for all $i \in [d]$. Hence $s_d^l \left(\prod_{t \in [d-1]} s_t^{l+k}\right) m = 0$. Now doing this inductively for each $i \in [d]$, we get $\left(\prod_{t \in [d]} s_t^{l}\right) m = 0$. Hence, the LHS is contained in the RHS. The other containment is clear, and this proves (i).

(ii) Since s_i^l divides x and y, $x - y = s_i^l z$ for some $z \in M$. Given that $s_i(x - y) = 0$. Hence $s_i^{l+1} z = 0$. By the definition of l, $s_i^l z = 0$. Therefore $x - y = 0$ which implies $x = y$. □

2.1. **The Artin-Rees Lemma and vanishing of maps on Tor modules.**

Lemma 2.3 (Artin-Rees)**.** *Let R denote a commutative Noetherian ring and I be an ideal in R. Let M be a finitely generated R-module and $N \subseteq M$. Then there exists $h \in \mathbb{N}$ such that for all $r \geq 0$*

$$I^{r+h} M \cap N = I^r (I^h M \cap N)$$

A weaker version of the Artin-Rees lemma which is often useful is

$$I^{r+h} M \cap N \subseteq I^r N.$$

Lemma 2.4. *Let $I \subseteq R$ be an ideal. Let $P_\bullet$ be a projective resolution of M. Let $B_i \subseteq P_i$ be the image of ∂_i^P. Then*

$$\operatorname{Tor}_i^R(M, R/I) = \operatorname{Tor}_1^R(P_{i-1}/B_{i-1}, R/I) = (IP_{i-1} \cap B_{i-1})/IB_{i-1}$$

Proof. From [Wei94, Exercise 2.4.3], $\operatorname{Tor}_i^R(M, -) = \operatorname{Tor}_1^R(Z_{i-2}, -)$ and since $P_\bullet$ is exact,

$$Z_{i-2} = B_{i-2} = P_{i-1}/Z_{i-1} = P_{i-1}/B_{i-1}.$$

This proves the first equality. Applying the long exact sequence of Tor to the exact sequence $0 \to B_{i-1} \to P_{i-1} \to P_{i-1}/B_{i-1} \to 0$ and noting that $\operatorname{Tor}_1^R(P_{i-1}, R/I) = 0$ yields

$$\begin{aligned}\operatorname{Tor}_1^R(P_{i-1}/B_{i-1}, R/I) &= \ker\left(B_{i-1} \otimes R/I \to P_{i-1} \otimes R/I\right)\\ &= \ker\left(B_{i-1}/IB_{i-1} \to P_{i-1}/IP_{i-1}\right)\\ &= (IP_{i-1} \cap B_{i-1})/IB_{i-1}\end{aligned}$$

□

Remark 2.5. By the Artin-Rees lemma, for all $i \geq 1$, there exists $h(i) \in \mathbb{N}$ such that for all $r \geq 0$,

$$I^{r+h(i)}P_{i-1} \cap B_{i-1} \subseteq I^r B_{i-1}. \tag{1}$$

By Lemma 2.4, the natural map

$$(I^{r+h(i)}P_{i-1} \cap B_{i-1})/I^{r+h(i)}B_{i-1} \to (I^r P_{i-1} \cap B_{i-1})/I^r B_{i-1}$$

is same as the natural map

$$\operatorname{Tor}_i^R(R/I^{r+h(i)}, M) \to \operatorname{Tor}_i^R(R/I^r, M).$$

The above map on Tor_i is zero if and only if $I^{r+h(i)}P_{i-1} \cap B_{i-1} \subseteq I^r B_{i-1}$, which is true from equation (1). This also shows that equation (1) is independent of the projective resolution.

As an immediate consequence, we get the following result.

Proposition 2.6. *Given an ideal $I \subseteq R$ and a finitely generated R-module M, for all $i \geq 1$, there exists $h(i) \in \mathbb{N}$ such that for all $r \geq 1$, the map $\operatorname{Tor}_i^R(R/I^{r+h(i)}, M) \to \operatorname{Tor}_i^R(R/I^r, M)$ is zero.*

In [EH05] and [AHS15], equation (1) is studied for free resolutions of the module M.

Definition 2.7. [AHS15] *Let R be a commutative Noetherian ring and M be a finitely generated module. Then M is said to be syzygetically Artin-Rees with respect to I if there exists a free resolution $P_\bullet$ and a uniform integer h exists such that for all $r \geq 0$ and for $i \geq 0$,*

$$I^{r+h}P_i \cap B_i \subseteq I^r B_i. \tag{2}$$

In [AHS15, Cor. 4.9], it is proved that any finitely generated module M is syzygetically Artin-Rees for any ideal I in a local ring which answers Question B in [EH05]. A similar Tor vanishing statement has been studied in [And74], which uses a weaker version of the Artin-Rees lemma.

2.2. **The intersection theorem and Cohen-Macaulay rings.** In this final subsection, we recall two well-known results for later use.

Theorem 2.8 ((Intersection theorem) : Peskine-Szpiro)**.** *Let R be a local ring and M and N be non-zero finitely generated modules over R. If $M \otimes_R N$ has finite length, then $dim(N) \leq pd_R(M)$.*

An immediate well-known consequence is the following criterion for a Noetherian local ring to be Cohen-Macaulay.

Corollary 2.9. *Let R be a Noetherian local ring. Then there exists a non-zero finitely generated R-module M of finite length and finite projective dimension if and only if R is Cohen-Macaulay.*

Proof. Take $N = R$ in the above theorem. Then $dim(R) \leq pd(M)$. Since M has finite length, $depth(M) = 0$. Applying the Auslander-Buchsbaum formula yields

$$dim(R) \leq pd(M) = depth(R) - depth(M) = depth(R).$$

Therefore, R is Cohen-Macaulay. The converse clearly holds. $\square$

2.3. **Filtrations of ideals and asymptotic behaviour of projective dimension.**

Definition 2.10. *A filtration $\{J_n\}$ of I, i.e. $I \supseteq J_1 \supseteq \cdots \supseteq J_{n-1} \supseteq J_n \supseteq \cdots$ where each J_n is an ideal in R, is said to be equivalent to the I-adic filtration $\{I^k\}$ if for all $k \in \mathbb{N}$, there exists n such that $J_n \subseteq I^k$ and for all $n \in \mathbb{N}$, there exists k such that $I^k \subseteq J_n$.*

Example 2.11.
(i) For an ideal I in R with a generating set $\underset{\sim}{s}$, we define $I_{\underset{\sim}{s}}^{[r]} := (\underset{\sim}{s}^r)$. Clearly $I_{\underset{\sim}{s}}^{[r]} \subseteq I^r$ and $I^{rd} \subseteq I_{\underset{\sim}{s}}^{[r]}$ where d denotes the number of elements in the sequence $\underset{\sim}{s}$. Hence, the square power filtration $\{I_{\underset{\sim}{s}}^{[n]}\}$ is equivalent to the power filtration $\{I^k\}$.
(ii) For a regular ring R and for an ideal $I \subseteq R$, the symbolic power filtration $\{I^{(n)}\}$ of I is known to be equivalent to $\{I^k\}$ ([Ver88, Thm. 3.5]). In fact, the hypothesis on R can be further weakened ([Ver88, Propn. 4.7]).

Remark 2.12. Let I be an ideal of R and $\underset{\sim}{s}$ be a generating set. If $\text{char}(R) = p > 0$ is prime, then $I_{\underset{\sim}{s}}^{[p^n]}$ is the n-th Frobenius power of I and hence is independent of the generating set $\underset{\sim}{s}$. We denote it by $I^{[p^n]}$.

The next lemma follows directly from [PS73, Theorem 1.7].

Lemma 2.13. *Let* $\text{char}(R) = p > 0$, *where p is a prime and $I \subseteq R$ be an ideal. If $pd(R/I) < \infty$ then $pd(R/I) = pd(R/I^{[p^n]})$ for every $n \in \mathbb{N}$.*

Definition 2.14. *Let $I \subseteq R$ be an ideal. The ideal I is said to have eventually finite projective dimension (efpd) if there is a filtration $\{J_n\}$ of I equivalent to $\{I^k\}$ such that $pd(R/J_n) < \infty$ for all $n \in \mathbb{N}$.*

Remark 2.15. The property of being efpd is shared by ideals with the same radical ideal i.e. if $I, J \subseteq R$ are ideals such that $\sqrt{I} = \sqrt{J}$, then I has efpd if and only if J has efpd.

We discuss a few classes of ideals which has eventually finite projective dimension.

Example 2.16.
(1) When R is a regular ring, every ideal has finite projective dimension and hence every ideal has efpd.
(2) Let $\text{char}(R) = p > 0$, where p is a prime. If $pd(R/I) < \infty$, then by Lemma 2.13, $pd(R/I) = pd(R/I^{[p^n]})$ for all $n \in \mathbb{N}$. Using the square-power filtration $\{I^{[p^n]}\}$ of I, we can conclude that I has efpd. Hence every ideal having finite projective dimension has efpd in prime characteristic.
(3) Let I be a set-theoretic complete intersection ideal i.e. there exists a regular sequence $a_1, \ldots, a_n$ such that $\sqrt{(a_1, \ldots, a_n)} = \sqrt{I}$. Since the Koszul complex on a regular sequence is a resolution and powers of elements in a regular sequence also form a regular sequence, $pd(R/(a_1^k, \ldots, a_n^k)) < \infty$ for all $k \in \mathbb{N}$. Hence every set-theoretic complete intersection ideal has efpd.
(4) As a special case of (3), the maximal ideal in a Cohen-Macaulay local ring has efpd.
(5) Every nilpotent ideal has efpd, since the filtration $\{I^n\}$ eventually becomes zero.

(6) Let R be an artinian ring with $Maxspec(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ for some $t \geq 1$. Then there exists $n \gg 0$ such that $R \cong R/\mathfrak{m}_1^n \times \cdots \times R/\mathfrak{m}_t^n$ and $(\mathfrak{m}_1 \cdots \mathfrak{m}_t)^n = 0$. We claim that every ideal $I \subseteq R$ has efpd. Without loss of generality, we may assume I is a radical ideal. Hence $I = \prod_{i\in\Lambda} \mathfrak{m}_i$ for some subset $\Lambda \subseteq [t]$. If R is a local ring, then I is the maximal ideal which is nilpotent. Hence I has efpd. Suppose there are more than one maximal ideal in R. Then we may assume $\Lambda \subset [t]$ to be any non-empty proper subset. Then for all $k \geq n$,
$$I^k = \prod_{i\in\Lambda} \mathfrak{m}_i^k \cong \prod_{j\in[t]\setminus\Lambda} R/\mathfrak{m}_j^k \cong \prod_{j\in[t]\setminus\Lambda} R/\mathfrak{m}_j^n$$
is a projective R-module. Hence I has efpd.

(7) Let $f : R \to S$ be a flat ring homomorphism and $I \subseteq R$ has efpd. Then there exists a filtration $\{J_n\}$ of I such that $pd(R/J_n) < \infty$ and $\{J_n\}$ is equivalent to $\{I^k\}$. Hence we get $pd(S/(J_nS)) < \infty$ and $\{J_nS\}$ is equivalent to $\{(IS)^k\}$. Thus $IS \subseteq S$ has efpd.

(8) If $I \subseteq R$ has efpd, then $I_\mathfrak{p} \subseteq R_\mathfrak{p}$ has efpd for all $\mathfrak{p} \in Spec(R)$, since localisation is flat.

(9) Let $R = k[x_1, x_2, x_3, x_4]_\mathfrak{m}/(x_1x_4 - x_2x_3)$ where $k = \mathbb{Z}/p\mathbb{Z}$ and $\mathfrak{m} = (x_1, x_2, x_3, x_4)$. The construction of Dutta-Hochster-Mclaughlin[DHM85] shows the existence of an R-module M of finite length and finite projective dimension such that the intersection multiplicity $\chi_R(M, N) < 0$ for some module N. If there is a flat local ring homomorphism from a regular local ring to R, that would imply that $\chi_R(M, N) \geq 0$. Therefore there is no flat local ring homomorphism from any regular local ring to R. Since R has a module of finite length and finite projective dimension, by Corollary 2.9, R is a Cohen-Macaulay local ring. Thus $\mathfrak{m}R \subseteq R$ has efpd. Thus we have an example of an ideal having efpd, which is not extended from a regular local ring under a flat map.

Later in section 4, we give a necessary and sufficient condition for a maximal ideal to have eventually finite projective dimension and in particular exhibit a class of ideals which do not have eventually finite projective dimension. We also give a necessary condition for arbitrary ideals to have eventually finite projective dimension.

2.4. **Subcategories of mod(R).** We denote by $mod(R)$ the category of finitely generated R-modules.

Definition 2.17. *(1) A Serre subcategory, denoted by $\mathscr{L}$, is a full subcategory of $mod(R)$ such that for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $mod(R)$, $M \in \mathscr{L}$ if and only if $M', M'' \in \mathscr{L}$.*

(2) A set $V \subseteq Spec(R)$ is called a specialization closed set if it is a union of closed sets.

(3) Given a specialization closed set V, there is a corresponding Serre subcategory defined as
$$\mathscr{L}_V = \{M \in mod(R) \mid Supp(M) \subseteq V\}.$$

(4) Given a Serre subcategory $\mathscr{L}$, there is a corresponding specialization closed set defined as
$$V_\mathscr{L} = \bigcup_{M\in\mathscr{L}} Supp(M).$$

Remark 2.18. Note that $\mathscr{L}_{V_\mathscr{L}} = \mathscr{L}$ and $V_{\mathscr{L}_V} = V$.

Definition 2.19. *A full subcategory $\mathcal{A} \subseteq mod(R)$ is said to be resolving if*

(1) R is in $\mathcal{A}$.

(2) $M \oplus N \in \mathcal{A}$ if and only if M and N are in $\mathcal{A}$.

(3) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence and $M'' \in \mathcal{A}$, then $M \in \mathcal{A}$ if and only if $M' \in \mathcal{A}$.

As a consequence, the category of finitely generated projective R-modules $\mathcal{P}$ is contained in any resolving subcategory. Resolving subcategories allow for generalizing the concept of projective dimension. Some examples of resolving subcategories are $\mathcal{P}, mod(R)$, the category of Gorenstein dimension zero modules (also called totally reflexive modules) and the category of maximal Cohen-Macaulay (MCM) modules in a Cohen-Macaulay local ring. Basic properties and statements about resolving subcategories may be found in [San17].

Definition 2.20. *Let $\mathcal{T} \subseteq mod(R)$ be a full subcategory. A full subcategory $\mathcal{T}$ is a thick subcategory of $mod(R)$ if it is resolving and for any exact sequence with $M', M, M'' \in mod(R)$,*

$$0 \to M' \to M \to M'' \to 0$$

$M, M' \in \mathcal{T}$ implies $M'' \in \mathcal{T}$.

Remark 2.21. Let $\Omega^i M$ denote the i^{th} syzygy of M. For a resolving subcategory $\mathcal{A}$, define the closure $\overline{\mathcal{A}} = \{M \in mod(R) \mid \Omega^{\gg 0} M \in \mathcal{A}\}$. Then $\overline{\mathcal{A}}$ is a thick subcategory of $mod(R)$.

Example 2.22. Here are some examples of thick subcategories of $mod(R)$ which arise as a closure of a resolving subcategory.

(1) $\overline{\mathcal{P}}$ is the category of finitely generated modules having finite projective dimension.
(2) The closure of Gorenstein dimension zero modules is the full subcategory of $mod(R)$ having finite Gorenstein dimension.
(3) When R is a Cohen-Macaulay local ring, $\overline{MCM(R)} = mod(R)$.

We fix the following notations: For $X_\bullet \in Ch^b(mod(R))$,

(1) $\min_c(X_\bullet) = sup\{n \mid P_\bullet = 0 \ \forall i < n\}$
(2) $\min(X_\bullet) = sup\{n \mid H_i(X_\bullet) = 0 \ \forall i < n\}$
(3) $\mathrm{Supph}(X_\bullet) = \{n \mid H_n(X_\bullet) \neq 0\}$
(4) $\mathrm{Width}(X_\bullet) = sup\{i - j \mid H_i(P_\bullet), H_j(X_\bullet) \neq 0\}$ if $X_\bullet$ is not acyclic and $\mathrm{Width}(X_\bullet) = 0$ if $X_\bullet$ is acyclic
(5) $\Sigma X_\bullet$ denotes the shift of the complex $X_\bullet$, that is, $(\Sigma X_\bullet)_k = (X_\bullet)_{k+1}$

2.5. **Preliminaries on K_0 of certain categories.** In [CS22], the stable range of K_0 relative to a Serre subcategory $\mathscr{L}$, denoted by $\alpha(\mathscr{L})$ was defined and studied. Also a computable invariant $\beta(\mathscr{L}) \geq \alpha(\mathscr{L})$ was introduced and studied. We recall below the definitions and results about $\beta(\mathscr{L})$ which are relevant to this article. Later in Section 3, we give bounds on $\beta(\mathscr{L})$ in certain cases.

Definition 2.23. *(1) For $P_\bullet \in Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$, a reducer of $P_\bullet$ is a pair $(Q_\bullet, u)$ where $Q_\bullet \in Ch^{[0,k-1]}_{\mathscr{L}}(\mathcal{P})$ and $u : Q_\bullet \to P_\bullet$ is a chain complex map such that $u_0 : Q_0 \to P_0$ is surjective.*
(2) $\beta(\mathscr{L}) = inf\{m \in \mathbb{N} \mid$ every $P_\bullet \in Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$ has a reducer, for all $k > m\}$

Recall the notation that $\underset{\sim}{s} = s_1, \ldots, s_d$.

Lemma 2.24. [CS22] *Let $I = (\underset{\sim}{s})$ be an ideal and $\mathscr{L} = \mathscr{L}_{V(I)}$.*

(1) When $\mathscr{L} \neq 0$, $grade(I) \leq \beta(\mathscr{L})$.
(2) For any $P_\bullet \in Ch^{[0,k]}_{\mathscr{L}}(\mathcal{P})$, there exists a chain complex map $\psi : K(\underset{\sim}{s}^r; P_0) \to P_\bullet$ for some $r \in \mathbb{N}$ such that ψ_0 is a surjection. When $k > d$, $(K(\underset{\sim}{s}^r), \psi)$ is a reducer of $P_\bullet$.

A direct consequence of [CS22, Theorem 6.3] is stated below.

Lemma 2.25. *If $grade(I) = \beta(\mathscr{L})$ then $K_0(\overline{\mathcal{P}} \cap \mathscr{L}) \cong K_0(D^b_{\mathscr{L}}(\mathcal{P}))$.*

Remark 2.26. By careful observation of the proof of Theorem 5.1. in [CS22], we can enhance the bound on $\beta(\mathscr{L}_{V(I)})$ in Theorem 5.1 in [CS22] as follows: Let $\{J_n\}$ be any filtration equivalent to $\{I^k\}$. Then

$$\beta(\mathscr{L}_{V(I)}) \leq \limsup \{pd(R/J_n) \mid n \in \mathbb{N}\}$$

When J_n is chosen to be the square-power filtration $I_{\underset{\sim}{s}}^{[r]}$, we get the bound in [CS22].
We also note that the RHS of the inequality is also an upper bound for the non-vanishing of the local cohomology $H_I^i(M)$ for an R-module M, since it is the direct limit of $Ext_R^i(R/J_n, M)$.

2.6. **Notations for further use.** We collect below some notations that will be used in the rest of the article.

- $e_{\underset{\sim}{i}} := e_{i_1} \wedge \cdots \wedge e_{i_m}$ where $\underset{\sim}{i} = \{i_1, \ldots, i_m\} \in \mathcal{P}_m^d$ with $i_j < i_{j+1}$.
- $P_{\underset{\sim}{i}}$: the R-linear map $\bigoplus\limits_{\underset{\sim}{i} \in \mathcal{P}_m^d} Me_{\underset{\sim}{i}} \to M$ where $\sum\limits_{\underset{\sim}{i} \in \mathcal{P}_m^d} c_{\underset{\sim}{i}} e_{\underset{\sim}{i}}$ maps to $c_{\underset{\sim}{i}}$.
- $\underset{\sim}{i} \sqcup k$: the $(m+1)$-ordered subset containing $\underset{\sim}{i}$ and k where $k \notin \underset{\sim}{i}$.
- $\eta^{n,k}$: the chain complex map from $K(\underset{\sim}{s}^n; M)$ to $K(\underset{\sim}{s}^k; M)$ where $\eta_m^{n,k}$ maps $e_{\underset{\sim}{i}}$ to $(s_{i_1} \cdots s_{i_m})^{n-k} e_{\underset{\sim}{i}}$ for all $\underset{\sim}{i} \in \mathcal{P}_m^d$ and $m \in [d]$ where $n \geq k > 0$.
- l : the common stabilization point for the submodules $(0 :_M s_i^t)$ (defined in notations 2.1).

Note that for $n \geq t \geq k$, $\eta^{n,k} = \eta^{t,k} \circ \eta^{n,t}$. Note that when $M = R$, $\eta^{n,k}$ is a map of dg algebras.

3. The Chain complex map from the Tate resolution to the Koszul complex

In this section, we define the Tate resolution and prove the main theorem about the existence of a map from a suitable Tate resolution to the Koszul complex as mentioned in the introduction. We begin with a lemma about the vanishing of maps on Koszul homologies.

Lemma 3.1. *There exists $h \in \mathbb{N}$ such that for all $r \geq 1$, the map*

$$H_i(\eta^{h+rd,r}) : H_i(K(\underset{\sim}{s}^{h+rd}; M)) \longrightarrow H_i(K(\underset{\sim}{s}^r; M))$$

is zero for all $i \geq 1$. For $i = 0$, the map is the natural surjection $M/(\underset{\sim}{s}^{h+rd})M \to M/(\underset{\sim}{s}^r)M$.

Proof. Let $S = R[X_1, \ldots, X_d]$ be the polynomial ring and $f : S \to R$ be the ring map mapping X_i to s_i. We denote the S-module structure on M through f by ${}_SM$. Then for all $h \geq 0$, we have the following commutative diagram where all the horizontal maps are isomorphisms and vertical maps are natural.

$$\begin{tikzcd}
\operatorname{Tor}_i^S(S/(X_1^{h+rd}, \ldots, X_d^{h+rd}), {}_SM) \arrow[r, "\cong"] \arrow[d] & H_i(K(X_1^{h+rd}, \ldots, X_d^{h+rd}; {}_SM)) \arrow[r, "\cong"] \arrow[d] & H_i(K(\underset{\sim}{s}^{h+rd}; M)) \arrow[d] \\
\operatorname{Tor}_i^S(S/(X_1^r, \ldots, X_d^r), {}_SM) \arrow[r, "\cong"] & H_i(K(X_1^r, \ldots, X_d^r; {}_SM)) \arrow[r, "\cong"] & H_i(K(\underset{\sim}{s}^r; M))
\end{tikzcd}$$

Since the Koszul complex is concentrated in degrees zero to d, it is sufficient to prove that for all $1 \leq i \leq d$, there exists $h \geq 0$ such that

$$\operatorname{Tor}_i^S(S/(X_1^{h+rd}, \ldots, X_d^{h+rd}), {}_SM) \to \operatorname{Tor}_i^S(S/(X_1^r, \ldots, X_d^r), {}_SM) \tag{3}$$

is the zero map. Since the natural surjection $S/(X_1^{h+rd}, \ldots, X_d^{h+rd}) \twoheadrightarrow S/(X_1^r, \ldots, X_d^r)$ factors through $S/(X_1, \ldots, X_d)^{h+rd} \twoheadrightarrow S/(X_1, \ldots, X_d)^{rd}$, the map in (3) is zero if the natural map $\mathrm{Tor}_i^S(S/(X_1,\dots,X_d)^{h+rd}, {}_SM) \to \mathrm{Tor}_i^S(S/(X_1,\dots,X_d)^{rd}, {}_SM)$ is zero. By Proposition 2.6, for each $1 \le i \le d$, there exists $h(i) \in \mathbb{N}$ such that for all $r \ge 1$, the map

$$\mathrm{Tor}_i^S(S/(X_1,\dots,X_d)^{h(i)+r}, {}_SM) \to \mathrm{Tor}_i^S(S/(X_1,\dots,X_d)^{r}, {}_SM)$$

is zero. Choosing $h = max\{h(i) \mid 1 \le i \le d\}$ and using the above for rd, we get that

$$\mathrm{Tor}_i^S(S/(X_1,\dots,X_d)^{h+r}, {}_SM) \to \mathrm{Tor}_i^S(S/(X_1,\dots,X_d)^{r}, {}_SM)$$

is the zero map for all $1 \le i \le d$. □

We fix the integer h obtained above for the rest of this section i.e. for all $r \ge 1$,

$$H_i(K(\underset{\sim}{s}^{h+rd};M)) \longrightarrow H_i(K(\underset{\sim}{s}^{r};M))$$

is the zero map for all $1 \le i \le d$. The following technical lemma about the divisibility of elements is used for our induction statement in Theorem 3.5.

Lemma 3.2. *Given $n \ge 1$, $r \ge 1$, there exists $v := q(n,r) = h + (n+r)d$ such that for all $m \ge 1$ and $x \in \ker\left(\partial_m^{K(\underset{\sim}{s}^v;M)}\right)$, there exists $y \in K_{m+1}(\underset{\sim}{s}^r;M)$ such that :*

- *(a) $\partial_{m+1}^{K(\underset{\sim}{s}^r;M)}(y) = \eta_m^{v,r}(x)$.*
- *(b) $P_{\underset{\sim}{j}}(y)$ is divisible by $\displaystyle\prod_{j\in \underset{\sim}{j}} s_j^n$ for all $\underset{\sim}{j} \in \mathcal{P}_{m+1}^d$.*

Proof. From Lemma 3.1, the natural map $H_m(\eta^{v,n+r}) : H_m(K(\underset{\sim}{s}^v;M)) \to H_m(K(\underset{\sim}{s}^{n+r};M))$ is zero for all $m \ge 1$. Therefore for all $x \in \ker\left(\partial_m^{K(\underset{\sim}{s}^v;M)}\right)$, we get $\eta_m^{v,n+r}(x) \in B_m(K(\underset{\sim}{s}^{n+r};M))$, where $B_m(K(\underset{\sim}{s}^{n+r};M))$ denotes $\mathrm{Im}(\partial_{m+1}^{K(\underset{\sim}{s}^{n+r};M)})$. Hence the commutative diagram below shows that $\eta_m^{v,r}(x) \in B_m(K(\underset{\sim}{s}^r;M))$.

$$\begin{tikzcd}
 & K_{m+1}(\underset{\sim}{s}^{n+r};M) \arrow[r, "\eta_{m+1}^{n+r,r}"] \arrow[d, two heads, "\partial_{m+1}^{K(\underset{\sim}{s}^{n+r};M)}"] & K_{m+1}(\underset{\sim}{s}^{r};M) \arrow[r, "P_{\underset{\sim}{i}}"] \arrow[d, two heads, "\partial_{m+1}^{K(\underset{\sim}{s}^{r};M)}"] & \left(\prod_{i\in\underset{\sim}{i}} s_i^{n-1}\right) M \cdot e_{\underset{\sim}{i}} \\
\ker\left(\partial_m^{K(\underset{\sim}{s}^v;M)}\right) \arrow[r, "\eta_m^{v,n+r}", "(Lemma\ 3.1)"'] & B_m(K(\underset{\sim}{s}^{n+r};M)) \arrow[r, "\eta_m^{n+r,r}"] & B_m(K(\underset{\sim}{s}^{r};M)) &
\end{tikzcd}$$

Since $\eta_m^{v,n+r}(x) \in B_m(K(\underset{\sim}{s}^{n+r};M))$, there exists $z = \displaystyle\sum_{\underset{\sim}{j}\in\mathcal{P}_{m+1}^d} c_{\underset{\sim}{j}} e_{\underset{\sim}{j}} \in K_{m+1}(\underset{\sim}{s}^{n+r};M)$ such that $\partial_{m+1}^{K(\underset{\sim}{s}^{n+r};M)}(z) = \eta_m^{v,n+r}(x)$. Let $y := \eta_{m+1}^{n+r,r}(z)$. Then $\partial_{m+1}^{K(\underset{\sim}{s}^{r};M)}(y) = \eta_m^{v,r}(x)$. Also,

$$y = \sum_{\underset{\sim}{j}\in\mathcal{P}_{m+1}^d} \left(\prod_{j\in\underset{\sim}{j}} s_j^n\right) c_{\underset{\sim}{j}} e_{\underset{\sim}{j}} \quad \text{and hence} \quad P_{\underset{\sim}{j}}(y) = \left(\prod_{j\in\underset{\sim}{j}} s_j^n\right) m_{\underset{\sim}{j}}.$$

□

Theorem 3.3. *Let $r \ge 1$ and $m \ge 1$ and $x = \displaystyle\sum_{\underset{\sim}{i}\in\mathcal{P}_m^d} x_{\underset{\sim}{i}} e_{\underset{\sim}{i}} \in \ker\left(\partial_m^{K(\underset{\sim}{s}^r;M)}\right)$. Suppose $x_{\underset{\sim}{i}}$ is divisible by $\displaystyle\prod_{i\in\underset{\sim}{i}} s_i^{l+n}$ for each $\underset{\sim}{i} \in \mathcal{P}_m^d$. Then there exists $y \in \ker\left(\partial_m^{K(\underset{\sim}{s}^{n+r};M)}\right)$ such that $x = \eta_m^{n+r,r}(y)$.*

Proof. Let $m = 1$. Then $x = \sum_{i=1}^{d} x_i e_i$ and $\partial_1^{K(\underset{\sim}{s}^r;M)}(x) = \sum_{i=1}^{d} s_i^r x_i = 0$. Since s_i^{l+n} divides x_i, there exists $c_i \in M$ such that $x_i = s_i^{l+n} c_i$ for all $i \in [d]$. Define $y = \sum_{i=1}^{d} s_i^l c_i e_i \in K_1(\underset{\sim}{s}^{n+r}; M)$. Then $\eta_1^{n+r,r}(y) = \sum_{i=1}^{d} s_i^{l+n} c_i e_i = x$ and $\partial_1^{K(\underset{\sim}{s}^{n+r};M)}(y) = \sum_{i=1}^{d} s_i^{l+n+r} c_i = \sum_{i=1}^{d} s_i^r x_i = 0$.

Consider the case $m \geq 2$. Since $x = \sum_{\underset{\sim}{i} \in \mathcal{P}_m^d} x_{\underset{\sim}{i}} e_{\underset{\sim}{i}} \in \ker\left(\partial_m^{K(\underset{\sim}{s}^r;M)}\right)$, we have

$$\sum_{\underset{\sim}{j} \in \mathcal{P}_{m-1}^d} \left(\sum_{k \in [d] \setminus \underset{\sim}{j}} (-1)^{\tau(k,\underset{\sim}{j})+1} s_k^r x_{(\underset{\sim}{j} \sqcup k)} \right) e_{\underset{\sim}{j}} = 0 \tag{4}$$

where $\tau(k, \underset{\sim}{j})$ denotes the position of k in the ordered set $\underset{\sim}{j} \sqcup k$. Hence for each $\underset{\sim}{j} \in \mathcal{P}_{m-1}^d$,

$$\sum_{k \in [d] \setminus \underset{\sim}{j}} (-1)^{\tau(k,\underset{\sim}{j})+1} s_k^r x_{(\underset{\sim}{j} \sqcup k)} = 0 \tag{5}$$

Since $\prod_{t \in \underset{\sim}{i}} s_t^{l+n}$ divides $x_{\underset{\sim}{i}}$, we get $x_{\underset{\sim}{i}} = \left(\prod_{t \in \underset{\sim}{i}} s_t^{l+n} \right) x'_{\underset{\sim}{i}}$ for some $x'_{\underset{\sim}{i}} \in M$. Thus for each $\underset{\sim}{j} \in \mathcal{P}_{m-1}^d$ and $k \in [d] \setminus \underset{\sim}{j}$, we obtain

$$\begin{aligned}
0 = \sum_{k \in [d] \setminus \underset{\sim}{j}} (-1)^{\tau(k,\underset{\sim}{j})+1} s_k^r x_{(\underset{\sim}{j} \sqcup k)} &= \sum_{k \in [d] \setminus \underset{\sim}{j}} (-1)^{\tau(k,\underset{\sim}{j})+1} s_k^r \left(\prod_{t \in (\underset{\sim}{j} \sqcup k)} s_t^{l+n} \right) x'_{(\underset{\sim}{j} \sqcup k)} \\
&= \left(\prod_{t \in \underset{\sim}{j}} s_t^l \right) \left(\prod_{t \in \underset{\sim}{j}} s_t^n \right) \sum_{k \in [d] \setminus \underset{\sim}{j}} (-1)^{\tau(k,\underset{\sim}{j})+1} s_k^{n+r} (s_k^l x'_{(\underset{\sim}{j} \sqcup k)}).
\end{aligned}$$

By Lemma 2.2, for all $n \geq 1$, $\left(0 :_M \left(\prod_{t \in \underset{\sim}{j}} s_t^{n+l} \right) \right) = \left(0 :_M \left(\prod_{t \in \underset{\sim}{j}} s_t^l \right) \right)$. Thus,

$$\left(\prod_{t \in \underset{\sim}{j}} s_t^l \right) \sum_{k \in [d] \setminus \underset{\sim}{j}} (-1)^{\tau(k,\underset{\sim}{j})+1} s_k^{n+r} (s_k^l x'_{(\underset{\sim}{j} \sqcup k)}) = 0.$$

Take $y_{\underset{\sim}{i}} = \left(\prod_{t \in \underset{\sim}{i}} s_t^l \right) x'_{\underset{\sim}{i}}$. Observe that $x_{\underset{\sim}{i}} = \left(\prod_{t \in \underset{\sim}{i}} s_t^n \right) y_{\underset{\sim}{i}}$. Define $y := \sum_{\underset{\sim}{i} \in \mathcal{P}_m^d} y_{\underset{\sim}{i}} e_{\underset{\sim}{i}} \in K_m(\underset{\sim}{s}^{n+r}; M)$. Then, for all $\underset{\sim}{j} \in \mathcal{P}_{m-1}^d$, the above equation yields

$$\sum_{k \in [d] \setminus \underset{\sim}{j}} (-1)^{\tau(k,\underset{\sim}{j})+1} s_k^{n+r} y_{(\underset{\sim}{j} \sqcup k)} = 0.$$

Therefore $y \in \ker\left(\partial_m^{K(\underset{\sim}{s}^{n+r};M)}\right)$. Clearly $\eta_m^{n+r,r}(y) = x$. $\square$

Definition-Remark 3.4. Let $\underset{\sim}{f} = f_1 \dots, f_n \in R$. Taking intuition from the construction of the Tate resolution (refer e.g. [Avr98]), we now construct a chain complex $T(\underset{\sim}{f}; M)$. Define

$$T_i(\underset{\sim}{f}; M) = \begin{cases} 0 & i < 0 \\ K_i(\underset{\sim}{f}; M) & i = 0, 1 \\ K_i(\underset{\sim}{f}; M) \oplus R^{t_i} & i \geq 2 \end{cases}$$

where the differentials and t_i are inductively defined as follows :
$\partial_1^{T(\underset{\sim}{f};M)} := \partial_1^{K(\underset{\sim}{f};M)}$. Choose a generating set S_1 of $Z_1(T(\underset{\sim}{f}; M))$ and define $t_2 = |S_1|$. Suppose for all $j \leq r-1$, the differentials $\partial_j^{T(\underset{\sim}{f};M)}$ have been defined , generating sets S_j of $Z_j(T(\underset{\sim}{f}; M))$ have been chosen and $t_{j+1} = |S_j|$. Define $\partial_r^{T(\underset{\sim}{f};M)}$ by $\partial_r^{T(\underset{\sim}{f};M)}|_{K_r(\underset{\sim}{f};M)} = \partial_r^{K(\underset{\sim}{f};M)}$ and $\partial_r^{T(\underset{\sim}{f};M)}$ maps the basis vectors of R^{t_r} to the set S_{r-1} of $Z_{r-1}(T(\underset{\sim}{f}; M))$. Choose a generating set S_r of $Z_r(T(\underset{\sim}{f}; M))$ and define $t_{r+1} = |S_r|$.

Note that $T(\underset{\sim}{f}; M)$ satisfies the following properties :

(1) $K(\underset{\sim}{f}; M)$ is a subcomplex of $T(\underset{\sim}{f}; M)$.
(2) $H_i(T(\underset{\sim}{f}; M)) = 0$ for all $i \geq 1$ and $H_0(T(\underset{\sim}{f}; M)) = M/(\underset{\sim}{f})M$.

We denote $T(\underset{\sim}{f}; R)$ by $T(\underset{\sim}{f})$. A slightly enhanced version of the above construction for $T(\underset{\sim}{f})$ in order to obtain a dg R-algebra structure on $T(\underset{\sim}{f})$ is precisely the Tate resolution on $R/(\underset{\sim}{f})$ as defined in [Avr98].

We are now set up to state and prove the main theorem.

Theorem 3.5. *Given $r \geq 1$, there exists a chain complex map $\phi : T(\underset{\sim}{s}^{u(r)}; M) \to K(\underset{\sim}{s}^r; M)$ with $\phi|_{K(\underset{\sim}{s}^{u(r)};M)} = \eta^{u(r),r}$ for $u(r) = (h + l)\left(\sum_{j=0}^{d-1} d^j\right) + rd^d$. In particular, there exists a chain complex map $\phi : T(\underset{\sim}{s}^{u(r)}) \to K(\underset{\sim}{s}^r)$ such that $\phi_0 = \eta_0^{u(r),r} = id_R$.*

Proof. Recall the notation $q(n, r) = h + (n + r)d$ from Lemma 3.2. Define

$$q^{(i)} = \begin{cases} q(0, r) - r = h + rd - r & i = 0 \\ q(q^{(i-1)} + l, r) - r & i \geq 1 \end{cases} .$$

Claim : For $i \geq 0$, $q^{(i)} = h\left(\sum_{j=0}^{i} d^j\right) + rd^{i+1} + l\left(\sum_{j=0}^{i-1} d^j\right) d - r$.

The claim is clearly true for the base case $i = 0$ and follows by induction as shown by the calculation below :

$$\begin{aligned} q^{(i+1)} = q(q^{(i)} + l, r) - r &= h + \left(h\left(\sum_{j=0}^{i} d^j\right) + rd^{i+1} + l\left(\sum_{j=0}^{i-1} d^j\right) d - r + l + r\right) d - r \\ &= h\left(\sum_{j=0}^{i+1} d^j\right) + rd^{i+2} + l\left(\sum_{j=0}^{i} d^j\right) d - r. \end{aligned}$$

Therefore $q^{(d-1)} + l + r = h\left(\sum_{j=0}^{d-1} d^j\right) + rd^d + l\left(\sum_{j=0}^{d-2} d^j\right) d + l = (h+l)\left(\sum_{j=0}^{d-1} d^j\right) + rd^d = u(r)$.

For the rest of this proof, we denote $u(r)$ by u. We now proceed to construct ϕ. For $m = 0$, define ϕ_0 to be the identity. For $m > d$, the zero map is the only possible map. For $1 \leq m \leq d$, we will inductively define ϕ_m such that

(a) $\phi_{m-1}\partial_m^{T(\underset{\sim}{s}^u;M)} = \partial_m^{K(\underset{\sim}{s}^r;M)}\phi_m$

(b) $\phi_m|_{K_m(\underset{\sim}{s}^u;M)} = \eta_m^{u,r}$

(c) $P_{\underset{\sim}{i}}\phi_m$ is divisible by $\prod_{i\in\underset{\sim}{i}} s_i^{q^{(d-m)}+l}$ for all $\underset{\sim}{i} \in \mathcal{P}_m^d$.

Then, we will show that the constructed map $P_{\underset{\sim}{i}}\phi_d$ being divisible by $\prod_{i\in\underset{\sim}{i}} s_i^{q^{(0)}+l}$ will imply that $\phi_d\partial_{d+1}^{T(\underset{\sim}{s}^u;M)} = 0 = \partial_{d+1}^{K(\underset{\sim}{s}^r;M)}\phi_{d+1}$, which will complete the proof.

In the base case $m = 1$, note that $T_1(\underset{\sim}{s}^u; M) = M^d \cong \bigoplus_{i=1}^{d} Me_i$ and so we can define ϕ_1 by extending $\phi_1(e_i) = s_i^{u-r}e_i$. Then $\phi_0\partial_1^{T(\underset{\sim}{s}^u;M)} = \partial_1^{K(\underset{\sim}{s}^r;M)}\phi_1$, $\phi_1|_{K_1(\underset{\sim}{s}^u;M)} = \eta_1^{u,r}$ and $P_i\phi_1$ is divisible by $s_i^{q^{(d-1)}+l}$, i.e. conditions (a), (b) and (c) hold.

Suppose ϕ_t exists for $1 \le t < m \le d$ such that the conditions (a), (b) and (c) hold. Recall that $T_m(\underset{\sim}{s}^u; M) = K_m(\underset{\sim}{s}^u; M) \oplus R^{t_m}$. We will define ϕ_m on each of the summands separately and check that (a), (b) and (c) above are satisfied on the summands. Define $\phi_m|_{K_m(\underset{\sim}{s}^u;M)} := \eta_m^{u,r}$. Since $\partial_m^{T(\underset{\sim}{s}^u;M)}|_{K_m(\underset{\sim}{s}^u;M)} = \partial_m^{K(\underset{\sim}{s}^u;M)}$ by definition and $\eta^{u,r}$ is a chain complex map, $\phi_{m-1}\partial_m^{T(\underset{\sim}{s}^u;M)}|_{K_m(\underset{\sim}{s}^u;M)} = \partial_m^{K(\underset{\sim}{s}^r;M)}\phi_m|_{K_m(\underset{\sim}{s}^u;M)}$. Since $P_{\underset{\sim}{i}}\eta_m^{u,r}$ is divisible by $\prod_{i\in\underset{\sim}{i}} s_i^{u-r}$ for all $\underset{\sim}{i} \in \mathcal{P}_m^d$ and $u - r \ge q^{(d-m)} + l$, the condition (c) follows for the summand $K_m(\underset{\sim}{s}^u; M)$. Let x be a basis element in the free R-module $R^{t_m} \subseteq T_m(\underset{\sim}{s}^u; M)$. By induction, $\phi_{m-1}\partial_m^{T(\underset{\sim}{s}^u;M)}(x) \in \ker\left(\partial_{m-1}^{K(\underset{\sim}{s}^r;M)}\right)$ and $P_{\underset{\sim}{i}}\phi_{m-1}\partial_m^{T(\underset{\sim}{s}^u;M)}(x)$ is divisible by $\prod_{i\in\underset{\sim}{i}} s_i^{q^{(d-m+1)}+l}$.
From Theorem 3.3, there exists $z \in \ker\left(\partial_{m-1}^{K(\underset{\sim}{s}^{q^{(d-m+1)}+r};M)}\right)$ such that $\eta_{m-1}^{q^{(d-m+1)}+r,r}(z) = \phi_{m-1}\partial_m^{T(\underset{\sim}{s}^u;M)}(x)$. Since $q^{(d-m+1)} + r = q(q^{(d-m)} + l, r)$ and $z \in \ker\left(\partial_{m-1}^{K(\underset{\sim}{s}^{q^{(d-m+1)}+r};M)}\right)$, Lemma 3.2 implies that there exists $y \in K_m(\underset{\sim}{s}^r; M)$ such that $\partial_m^{K(\underset{\sim}{s}^r;M)}(y) = \eta_{m-1}^{q^{(d-m+1)}+r,r}(z)$ and $P_{\underset{\sim}{i}}(y)$ is divisible by $\prod_{i\in\underset{\sim}{i}} s_i^{q^{(d-m)}+l}$. Define $\phi_m(x) := y$. Then,

$$\phi_{m-1}\partial_m^{T(\underset{\sim}{s}^u;M)}(x) = \eta_{m-1}^{q^{(d-m+1)}+r,r}(z) = \partial_m^{K(\underset{\sim}{s}^r;M)}(y) = \partial_m^{K(\underset{\sim}{s}^r;M)}\phi_m(x).$$

Extend this to a map ϕ_m on R^{t_m} by linearity. Since for every basis element of R^{t_m}, the properties (a) and (c) are satisfied, both of these hold for every element in R^{t_m}. Thus ϕ_m has been constructed inductively for $1 \le m \le d$.

It remains to check that $\phi_d\partial_{d+1}^{T(\underset{\sim}{s}^u;M)} = 0$. Let $x \in T_{d+1}(\underset{\sim}{s}^u; M)$. Since $P_{\underset{\sim}{i}}\phi_d\partial_{d+1}^{T(\underset{\sim}{s}^u;M)}(x)$ is divisible by $\prod_{i\in\underset{\sim}{i}} s_i^{q^{(0)}+l}$, by Theorem 3.3, there exists $z \in \ker\left(\partial_d^{K(\underset{\sim}{s}^{q^{(0)}+r};M)}\right)$ such that $\eta_d^{q^{(0)}+r,r}(z) = \phi_d\partial_{d+1}^{T(\underset{\sim}{s}^u;M)}(x)$. Since $q^{(0)} + r = q(0, r)$ and $z \in \ker\left(\partial_d^{K(\underset{\sim}{s}^{q^{(0)}+r};M)}\right)$, Lemma 3.2(a) implies that $\eta_d^{q^{(0)}+r,r}(z) \in \operatorname{Im}\left(\partial_{d+1}^{K(\underset{\sim}{s}^r;M)}\right) = 0$. This completes the proof. □

Remark 3.6. Note that in the previous Theorem 3.5, the obtained ϕ need not be unique since at each degree, there could be several choices of lifts. Since both the Tate resolution $T(\underset{\sim}{s}^{u(r)})$ and the Koszul complex $K(\underset{\sim}{s}^r)$ are dg R-algebras, one may ask if we could construct the above map such that it is a dg R-algebra map. Begin with any map $\phi : T(\underset{\sim}{s}^{u(r)}) \to K(\underset{\sim}{s}^r)$ obtained from Theorem 3.5. Since the dg R-algebra $T(\underset{\sim}{s}^{u(r)})$ is a polynomial algebra of the form $R[\{X_{ij} \mid i > 0, j \in \Lambda_i\}]$ where X_{ij} are variables of degree i, we can define a dg R-algebra map $\gamma : T(\underset{\sim}{s}^{u(r)}) \to K(\underset{\sim}{s}^r)$ by mapping $\gamma(X_{ij}) = \phi(X_{ij})$ and then extend it to all of $T(\underset{\sim}{s}^{u(r)})$ using the dg-structure. Hence there exists a map of dg R-algebras from $T(\underset{\sim}{s}^{u(r)})$ to $K(\underset{\sim}{s}^r)$ which extends $\eta^{u(r),r}$.

Corollary 3.7. *Let $r \geq 1$ and $u(r)$ denote the integer as in Theorem 3.5. Let $P_\bullet(M/(\underset{\sim}{s}^{u(r)})M)$ be a projective resolution of $M/(\underset{\sim}{s}^{u(r)})M$. Then there exists a chain complex map*

$$\xi : P_\bullet(M/(\underset{\sim}{s}^{u(r)})M) \to K(\underset{\sim}{s}^r; M)$$

such that $H_0(\xi)$ is the natural surjection $M/(\underset{\sim}{s}^{u(r)})M \twoheadrightarrow M/(\underset{\sim}{s}^r)M$.

Proof. By the lifting property for a complex of projective modules [Wei94, Theorem 2.2.6], there exists a chain complex map $\psi : P_\bullet(M/(\underset{\sim}{s}^{u(r)})M) \to T(\underset{\sim}{s}^{u(r)}; M)$ such that $H_0(\psi)$ is the identity map on $M/(\underset{\sim}{s}^{u(r)})M$. Composing ψ with the map $\phi : T(\underset{\sim}{s}^{u(r)}; M) \to K(\underset{\sim}{s}^r; M)$ obtained from Theorem 3.5, we get $\xi = \phi \circ \psi : P_\bullet(M/(\underset{\sim}{s}^{u(r)})M) \to K(\underset{\sim}{s}^r; M)$. Since $H_0(\phi)$ is the natural surjection $M/(\underset{\sim}{s}^{u(r)})M \twoheadrightarrow M/(\underset{\sim}{s}^r)M$, so is $H_0(\xi)$. $\square$

We now show that the vanishing of Koszul homologies is equivalent to the existence of the map in Theorem 3.5.

Remark 3.8. The picture below summarizes the structure of the proofs thus far.

Artin-Rees lemma $\Rightarrow$ vanishing of Tor maps $\Rightarrow$ vanishing of Koszul maps $\Rightarrow$ existence of ϕ

Note that once we know the existence of the map ϕ in Theorem 3.5, it follows immediately that $H_i(\eta^{u(r),r}) : H_i(K(\underset{\sim}{s}^{u(r)}; M)) \to H_i(K(\underset{\sim}{s}^r; M))$ is zero for all i since $\eta^{u(r),r}$ factors through $T(\underset{\sim}{s}^{u(r)}; M)$ which has zero homologies for all $i \neq 0$. Thus, the final implication in the picture above also goes the other way.

It is well-known (e.g. refer to the proof of Theorem 3.1) that the Koszul homology modules can be described in terms of Tor modules, which yields the second implication above. The next proposition and theorem show that the vanishing of maps between Tor modules can be obtained from the vanishing of maps of Koszul homology modules i.e. the second implication can also be reversed.

Proposition 3.9. *Let $I = (\underset{\sim}{s})$ be an ideal. Let $i \geq 1$ and $r \geq 1$. Then the following are equivalent.*

- *(i) There exists $v(i, r) \geq r$ such that the map $\operatorname{Tor}_i^R(R/I^{v(i,r)}, M) \to \operatorname{Tor}_i^R(R/I^r, M)$ is zero.*
- *(ii) There exists $w(i, r) \geq r$ such that $H_i(\eta) : H_i(K(\underset{\sim}{s}^{w(i,r)}; M)) \to H_i(K(\underset{\sim}{s}^r; M))$ is zero.*

Proof. (1) $\implies$ (2) : Let $r \geq 1$ and $i \geq 1$ be fixed. By applying Theorem 3.5, there exists a chain complex map $T(\underset{\sim}{s}^{u(r)}) \to K(\underset{\sim}{s}^r)$ for some $u(r) \in \mathbb{N}$. Applying (1) for $u(r)d \in \mathbb{N}$, we get $v(i, u(r)d) \geq u(r)d$ such that the map

$$\operatorname{Tor}_i^R(R/I^{v(i,u(r)d)}, M) \to \operatorname{Tor}_i^R(R/I^{u(r)d}, M) \tag{6}$$

is zero. We have natural surjections

$$R/(\underset{\sim}{s}^{v(i,u(r)d)}) \twoheadrightarrow R/I^{v(i,u(r)d)} \twoheadrightarrow R/I^{u(r)d} \twoheadrightarrow R/(\underset{\sim}{s}^{u(r)}).$$

Also note that $K(\underset{\sim}{s}^{v(i,u(r)d)}) \subseteq T(\underset{\sim}{s}^{v(i,u(r)d)})$. Thus we get natural maps

$$H_i(K(\underset{\sim}{s}^{v(i,u(r)d)}); M) \longrightarrow \operatorname{Tor}_i^R(R/(\underset{\sim}{s}^{v(i,u(r)d)}), M) \longrightarrow \operatorname{Tor}_i^R(R/I^{v(i,u(r)d)}, M) \longrightarrow$$
$$\operatorname{Tor}_i^R(R/I^{u(r)d}, M) \longrightarrow \operatorname{Tor}_i^R(R/(\underset{\sim}{s}^{u(r)}), M) \longrightarrow H_i(K(\underset{\sim}{s}^r); M)$$

Since the map in (6) is zero, we get the above composition to be zero. Thus defining $w(i, r) = v(i, u(r)d)$, the proof is complete.

(2) $\implies$ (1) : Given $w(i,r) \in \mathbb{N}$, by Theorem 3.5, there exists $u(w(i,r)) \geq w(i,r)$ such that there exists a chain complex map $T(\underset{\sim}{s}^{u(w(i,r))}) \to K(\underset{\sim}{s}^{w(i,r)})$. We have natural surjections

$$R/I^{u(w(i,r))d} \twoheadrightarrow R/(\underset{\sim}{s}^{u(w(i,r))}) \twoheadrightarrow R/(\underset{\sim}{s}^{r}) \twoheadrightarrow R/I^{r}$$

which induces the maps

$$\operatorname{Tor}_i^R(R/I^{u(w(i,r))d}, M) \longrightarrow \operatorname{Tor}_i^R(R/(\underset{\sim}{s}^{u(w(i,r))}), M) \longrightarrow \operatorname{Tor}_i^R(R/(\underset{\sim}{s}^{r}), M) \longrightarrow \operatorname{Tor}_i^R(R/I^{r}, M)$$

We will prove that the second map is zero and hence so is the composition. Since we have the chain complex maps $T(\underset{\sim}{s}^{u(w(i,r))}) \longrightarrow K(\underset{\sim}{s}^{w(i,r)}) \longrightarrow K(\underset{\sim}{s}^{r}) \longrightarrow T(\underset{\sim}{s}^{r})$ whose homology map at zeroth level are natural surjections, the natural map $\operatorname{Tor}_i^R(R/(\underset{\sim}{s}^{u(w(i,r))}), M) \to \operatorname{Tor}_i^R(R/(\underset{\sim}{s}^{r}), M)$ factors through $H_i(K(\underset{\sim}{s}^{w(i,r)}; M)) \to H_i(K(\underset{\sim}{s}^{r}; M))$ which is zero by assumption. Hence by defining $v(i,r) = u(w(i,r))d$, the claim follows. □

In [AHS15, Cor. 4.9], it is proved that when R is a Noetherian local ring, then M is syzygetically Artin-Rees with respect to I, i.e. a linear function $w(r)$ exists such that $\operatorname{Tor}_i^R(R/I^{w(r)}, M) \to \operatorname{Tor}_i^R(R/I^{r}, M)$ is zero for all $i \geq 1$. The next theorem provides a polynomial function $w(r)$, even for rings which are not necessarily local, such that the maps between Tor modules of degrees $w(r)$ to r vanish. Note that the function $w(r)$ is independent of the degree i, which enhances the degree-wise vanishing of maps between Tor modules in [And74, Chapter 10, Propn. 10 and Lemma 11].

Theorem 3.10. *Let R be a Noetherian ring and let $I = (\underset{\sim}{s})$ be an ideal and M be a finitely generated module. Given $r \geq 1$, there exists $w(r) := u(h + rd)$ such that the map*

$$\operatorname{Tor}_i^R(R/I^{w(r)}, M) \to \operatorname{Tor}_i^R(R/I^{r}, M)$$

is zero for all $i \geq 1$.

Proof. By Lemma 3.1, the map $H_i(\eta^{h+rd,r}) : H_i(K(\underset{\sim}{s}^{h+rd}; M)) \to H_i(K(\underset{\sim}{s}^{r}; M))$ is zero for all $i > 0$. By Theorem 3.5, there exists a chain complex map $T(\underset{\sim}{s}^{u(h+rd)}) \to K(\underset{\sim}{s}^{h+rd})$. Define $w(r) := u(h + rd)$. Thus we get a map of chain complexes

$$T(\underset{\sim}{s}^{w(r)}) \otimes M \to K(\underset{\sim}{s}^{h+rd}; M) \to K(\underset{\sim}{s}^{r}; M) \to T(\underset{\sim}{s}^{r}) \otimes M.$$

Since the map on homologies of the middle map is zero, we get

$$\operatorname{Tor}_i^R(R/I^{w(r)}, M) \to \operatorname{Tor}_i^R(R/I^{r}, M)$$

is the zero map for all $i > 0$. □

Corollary 3.11. *Let I be a principal ideal and M be a finitely generated module. Then the map $\operatorname{Tor}_i^R(R/I^{2h+l+r}, M) \to \operatorname{Tor}_i^R(R/I^{r}, M)$ is zero for all $i > 0$. Thus M is syzygetically Artin-Rees with respect to I.*

Proof. When I is a principal ideal, $d = 1$. So we get $w(r) = u(h + r) = 2h + l + r$ and hence the claim. □

Lemma 3.12. *Let $\operatorname{char}(R) = p > 0$, where p is a prime and $I = (\underset{\sim}{s})$ be an ideal in R. Then*

(1) For any $k > pd(R/I)$ and $P_\bullet \in Ch_{\mathscr{L}}^{[0,k]}(\mathcal{P})$, $T(\underset{\sim}{s}^{p^n}) \otimes P_0$ is a reducer of $P_\bullet$ for all $n \gg 0$.

(2) $\beta(\mathscr{L}_{V(I)}) \leq pd(R/I)$.

Proof. From Lemma 2.24 (2), there exists a chain complex map $\psi : K(\underset{\sim}{s}^{r}; M) \to P_\bullet$ for some $r \in \mathbb{N}$. By composing with the map $\phi : T(\underset{\sim}{s}^{u}) \to K(\underset{\sim}{s}^{r})$ constructed in Theorem 3.5, we get a chain complex map $\psi \circ \phi : T(\underset{\sim}{s}^{u}) \otimes P_0 \to P_\bullet$ for all large $u \in \mathbb{N}$ such that $(\psi \circ \phi)_0$ is surjective. From Lemma 2.13, $pd(R/I) = pd(R/I^{[p^n]})$ for all $n \in \mathbb{N}$. Hence whenever $k > pd(R/I) = pd(R/I^{[p^n]})$, $T(\underset{\sim}{s}^{p^n})$ is a reducer of $P_\bullet$ for all $n \gg 0$. Now from definition, $\beta(\mathscr{L}_{V(I)}) \leq pd(R/I)$. □

Theorem 3.13. *Let* $\operatorname{char}(R) = p > 0$*, where* p *is a prime and* $\mathscr{L} = \mathscr{L}_{V(I)}$ *where* I *be a perfect ideal in* R*. Then* $K_0(\overline{\mathcal{P}} \cap \mathscr{L}) \cong K_0(D^b_{\mathscr{L}}(\mathcal{P}))$.

Proof. By Lemma 2.24 and Lemma 3.12, $grade(I) \le \beta(\mathscr{L}) \le pd(R/I)$. Since I is a perfect ideal, $pd(R/I) = grade(I)$. Therefore $\beta(\mathscr{L}) = grade(I)$. Applying Lemma 2.25, we get $K_0(\overline{\mathcal{P}} \cap \mathscr{L}) \cong K_0(D^b_{\mathscr{L}}(\mathcal{P}))$. $\square$

4. A Derived Equivalence

The main results in this section largely follow along the lines of [SS17, Sections 3 and 4]. As mentioned in the introduction, the proofs go through with the role of the Koszul resolution in that article being played by the Tate resolution here. Unfortunately, the statements (specifically the crucial [SS17, Theorems 3.1 and 3.3]) there do not allow for direct application to our context since they use the structure of the Koszul resolution in the proofs.

We first define a notion called a strong reducer, which essentially collects the properties of the Koszul resolution used in the proofs in [SS17, Sections 3 and 4].

Definition 4.1. *Let* $\mathscr{L}$ *be a Serre subcategory,* $X_\bullet \in Ch^b_{\mathscr{L}}(mod(R))$ *and* $J \subsetneq R$ *such that* $X_\bullet$ *is not exact and* $R/J \in \mathscr{L}$*. We say that* $(T_\bullet, \alpha, I)$ *is a strong reducer of* $(X_\bullet, J)$ *if the following conditions are satisfied :*

(1) $T_\bullet \in Ch^b_{\mathscr{L}}(\mathcal{P})$
(2) $\min_c(T_\bullet) = m$ *where* $m = min(X_\bullet)$
(3) $\alpha : T_\bullet \to X_\bullet$ *is a chain complex map*
(4) $\operatorname{Supph}(T_\bullet) = \{m\}$ *and* $H_m(T_\bullet) = T_m/IT_m$
(5) $H_m(\alpha) : H_m(T_\bullet) \to H_m(X_\bullet)$ *is surjective.*
(6) $I \subseteq J$

A Serre subcategory $\mathscr{L}$ *is defined to have the strong reducer (SR) property, if for every* $X_\bullet \in Ch^b_{\mathscr{L}}(mod(R))$ *and* $J \subsetneq R$ *such that* $X_\bullet$ *is not an exact complex and* $R/J \in \mathscr{L}$*,* $(X_\bullet, J)$ *has a strong reducer.*

The following lemma is analogous to [SS17, Lemma 3.2].

Lemma 4.2. *Let* $\mathscr{L}$ *be a Serre subcategory,* $X_\bullet \in Ch^b_{\mathscr{L}}(mod(R))$ *and* $J \subsetneq R$ *such that* $R/J \in \mathscr{L}$*. Suppose* $(X_\bullet, J)$ *has a strong reducer* $(T_\bullet, \alpha, I)$*. Extend the chain complex map* α *to an exact triangle in* $D^b_{\mathscr{L}}(mod(R))$ *:*

$$\Sigma^{-1}C_\bullet \to T_\bullet \xrightarrow{\alpha} X_\bullet \to C_\bullet.$$

If $\operatorname{Width}(X_\bullet) > 0$*, then we have the following:*

(1) $\operatorname{Width}(C_\bullet) < \operatorname{Width}(X_\bullet)$
(2) $\operatorname{Width}(\Sigma^{-1}C_\bullet \oplus T_\bullet) < \operatorname{Width}(X_\bullet)$

Proof. Let $m = \min(X_\bullet)$. Since $H_i(T_\bullet) = 0$ for all $i \neq m$, we have $H_i(X_\bullet) \cong H_i(C_\bullet)$ for all $i \neq m, m+1$. Since $H_m(\alpha)$ is surjective, $H_m(C_\bullet) = 0$. Hence the claim follows. $\square$

The following lemma is similar to [SS17, Theorem 3.3 and Lemma 3.4]. The proof is really a check that the proofs of those results work with the Koszul resolution and choices of ideals in those proofs being replaced by a strong reducer. That being the case, we only give a sketch of the proof.

Lemma 4.3. *Suppose* $\mathcal{T}$ *is a thick subcategory of* $mod(R)$ *and* $\mathscr{L}$ *has the SR property. Let* $X_\bullet \xrightarrow{f} Y_\bullet$ *be a morphism in* $D^b_{\mathscr{L}}(\mathcal{T})$ *such that* $X_\bullet$ *and* $Y_\bullet$ *are chain complexes in* $Ch^b(\mathcal{T} \cap \mathscr{L})$*,* $\min(X_\bullet \oplus Y_\bullet) = m$ *and* $\min_c(X_\bullet), \min_c(Y_\bullet) \ge m$*. Then there exists chain complexes* $M^X_\bullet$ *and* $M^Y_\bullet$ *in* $Ch^b(\mathcal{T} \cap \mathscr{L})$ *and maps of chain complexes* $M^X_\bullet \xrightarrow{\beta^X} X_\bullet$*,* $M^Y_\bullet \xrightarrow{\beta^Y} Y_\bullet$ *and* $M^X_\bullet \xrightarrow{\kappa} M^Y_\bullet$*, such that*

- $M^X_\bullet, M^Y_\bullet \in Ch^b(\mathcal{T} \cap \mathscr{L})$ *with* $M^X_i = M^Y_i = 0$ *for all* $i \neq m$.
- *there is a commutative square in* $D^b_{\mathscr{L}}(\mathcal{T})$:

$$\begin{tikzcd} M^X_\bullet \arrow[r, "\beta^X"] \arrow[d, "\kappa"] & X_\bullet \arrow[d, "f"] \\ M^Y_\bullet \arrow[r, "\beta^Y"] & Y_\bullet \end{tikzcd}$$

- $H_m(\beta^X)$ *and* $H_m(\beta^Y)$ *are surjective.*

Let $C^X_\bullet$ *and* $C^Y_\bullet$ *be the cones of* β^X *and* β^Y *respectively. If* $\mathrm{Width}(X_\bullet \oplus Y_\bullet) = k > 0$, *then*

(1) $\mathrm{Width}(C^X_\bullet \oplus C^Y_\bullet) < k$
(2) $\mathrm{Width}(C^X_\bullet \oplus Y_\bullet) \leq k$
(3) If $\min(X_\bullet) < \min(Y_\bullet)$, *then* $\min(C^X_\bullet \oplus Y_\bullet) \leq k - 1$.

Proof. Define the complex $M^Y_\bullet$ to be $\Sigma^m Y_m$ and $\beta^Y : M^Y_\bullet \to Y_\bullet$ be the inclusion. Then $M^Y_\bullet \in Ch^b(\mathcal{T} \cap \mathscr{L})$ and β^Y is a chain complex map such that $H_m(\beta^Y)$ is surjective.
Let f be given by a roof diagram $X_\bullet \xleftarrow{q} Q_\bullet \xrightarrow{g} Y_\bullet$ where q is a quasi-isomorphism. We may assume $\min_c(Q_\bullet) = m$. Let $Q'_\bullet$ be the pullback of β^Y and g in $Ch^b(mod(R))$:

$$\begin{tikzcd} Q'_\bullet \arrow[r, dotted, "\nu"] \arrow[d, dotted, "\mu"] & Q_\bullet \arrow[d, "g"] \\ M^Y_\bullet \arrow[r, "\beta^Y"] & Y_\bullet \end{tikzcd}$$

Thus we get an exact sequence $0 \to Q'_\bullet \to Q_\bullet \oplus M^Y_\bullet \to B_\bullet \to 0$ where $B_\bullet$ is the image of the map $Q_\bullet \oplus M^Y_\bullet \xrightarrow{(g, -\beta^Y)} Y_\bullet$. We observe $B_\bullet \in Ch^b(\mathscr{L})$ and hence $B_\bullet \in Ch^b_{\mathscr{L}}(mod(R))$. Since $M^Y_\bullet$ and $Q_\bullet$ also lie in $Ch^b_{\mathscr{L}}(mod(R))$, we get $Q'_\bullet \in Ch^b_{\mathscr{L}}(mod(R))$. Set $\lambda = q \circ \nu$ and $J = Ann(X_m)$. Then $R/J \in \mathscr{L}$. Since $\mathscr{L}$ has the SR property, $(Q'_\bullet, J)$ has a strong reducer $(T_\bullet, \alpha, I)$. Therefore, we have the following commutative diagram in $D^b_{\mathscr{L}}(mod(R))$:

$$\begin{tikzcd} T_\bullet \arrow[r, "\alpha"] & Q'_\bullet \arrow[r, "\lambda"] \arrow[d, "\mu"] & X_\bullet \arrow[d, "f"] \\ & M^Y_\bullet \arrow[r, "\beta^Y"] & Y_\bullet \end{tikzcd}$$

Define $M^X_\bullet$ to be the chain complex $\Sigma^m H_m(T_\bullet) \in Ch^b(\mathcal{T} \cap \mathscr{L})$. Since $I \subseteq J = Ann(X_m)$ and $H_m(T_\bullet) \cong T_m/IT_m$, the map $\lambda_m \circ \alpha_m : T_m \to X_m$ factors through T_m/IT_m. Therefore, we get a chain complex map $\beta^X : M^X_\bullet \to X_\bullet$. Also $\mu \circ \alpha$ factors through $M^X_\bullet$ giving a chain complex map $\kappa : M^X_\bullet \to M^Y_\bullet$ such that the following diagram commutes in $D^b_{\mathscr{L}}(\mathcal{T})$:

$$\begin{tikzcd} M^X_\bullet \arrow[r, "\beta^X"] \arrow[d, "\kappa"] & X_\bullet \arrow[d, "f"] \\ M^Y_\bullet \arrow[r, "\beta^Y"] & Y_\bullet. \end{tikzcd}$$

To show that $H_m(\beta^X)$ is surjective, it is enough to show that $H_m(\nu)$ is surjective, which follows because $Q'_\bullet$ is just the degree-wise pullback. The statements about the Width of the complexes can be derived from the long exact sequence of homologies arising from the mapping cone, similar to the proof of the previous lemma. $\square$

We now state the technical heart of this section which is analogous to the key theorem [SS17, Theorem 4.5] of [SS17]. Let $\iota : D^b(\mathcal{T}\cap\mathcal{L}) \rightsquigarrow D^b_{\mathcal{L}}(\mathcal{T})$ be the natural functor induced by the inclusion map $Ch^b(\mathcal{T}\cap\mathcal{L}) \hookrightarrow Ch^b_{\mathcal{L}}(\mathcal{T})$.

Theorem 4.4. *Suppose $\mathcal{T}$ is a thick subcategory of mod(R) and $\mathcal{L}$ has the SR property. Then the functor $\iota : D^b(\mathcal{T}\cap\mathcal{L}) \rightsquigarrow D^b_{\mathcal{L}}(\mathcal{T})$ is an equivalence of categories.*

Proof. Since ι is a triangulated functor between triangulated categories, by [SS17, Lemma 2.19] it suffices to prove that ι is faithful on objects, full and essentially surjective. If $\iota(X_\bullet) = 0$, i.e. it is acyclic, then $X_\bullet$ is also acyclic. Hence ι is faithful on objects and it suffices to prove that ι is essentially surjective and full. This proof is verbatim the same as that of [SS17, Proposition 4.4] and hence, we only briefly mention the main steps in the proof and request the interested reader to read the details from [SS17]. The following induction statement on k will show that ι is essentially surjective and full :

(1) For every $P_\bullet \in D^b_{\mathcal{L}}(\mathcal{T})$ with Width$(P_\bullet) = k$, there exists $\tilde{P}_\bullet \in D^b(\mathcal{T}\cap\mathcal{L})$ such that $\iota(\tilde{P}_\bullet) \cong P_\bullet$.
(2) $Hom_{D^b(\mathcal{T}\cap\mathcal{L})}(X_\bullet, Y_\bullet) \to Hom_{D^b_{\mathcal{L}}(\mathcal{T})}(X_\bullet, Y_\bullet)$ is surjective for all $X_\bullet, Y_\bullet \in D^b(\mathcal{T}\cap\mathcal{L})$ such that Width$(X_\bullet \oplus Y_\bullet) = k$.

The base case is when $k = 0$. If Width$(P_\bullet) = 0$, then Supph$(P_\bullet) \subseteq \{m\}$ for some $m \in \mathbb{Z}$. Thus $P_\bullet$ is quasi-isomorphic to $\Sigma^m H_m(P_\bullet)$. It follows from the definitions of $\mathcal{T}$ and $\mathcal{L}$ that $H_m(P_\bullet) \in \mathcal{T}\cap\mathcal{L}$ and hence $\Sigma^m H_m(P_\bullet) \in D^b(\mathcal{T}\cap\mathcal{L})$, which proves (1). Similarly Width$(X_\bullet \oplus Y_\bullet) = 0$ implies that $X_\bullet$ and $Y_\bullet$ are quasi-isomorphic to $\Sigma^m H_m(X_\bullet)$ and $\Sigma^m H_m(Y_\bullet)$ for some $m \in \mathbb{Z}$ respectively. Note that for any R-modules M and N, regarding them as complexes, we have

$$Hom_{\mathcal{T}\cap\mathcal{L}}(M,N) \xrightarrow{\sim} Hom_{D^b(\mathcal{T}\cap\mathcal{L})}(M,N) \xrightarrow{\sim} Hom_{D^b_{\mathcal{L}}(\mathcal{T})}(\iota(M),\iota(N)) \xrightarrow{\sim} Hom_R(M,N).$$

Hence (2) follows when $k = 0$.

Assume the statement holds for all $k' < k$. Since $\mathcal{L}$ has the SR property, there exists $Q_\bullet \in Ch^b_{\mathcal{L}}(\mathcal{T})$ with Supph$(Q_\bullet) = \{m\}$ and an exact triangle $\Sigma^{-1}C_\bullet \xrightarrow{\alpha} Q_\bullet \xrightarrow{\beta} P_\bullet \xrightarrow{\gamma} C_\bullet$ with Width$(C_\bullet) < k$ and Width$(\Sigma^{-1}C_\bullet \oplus Q_\bullet) < k$. Applying the induction hypothesis (1) and (2), there exists $\iota(\tilde{C}_\bullet) = C_\bullet$ and the map

$$Hom_{D^b(\mathcal{T}\cap\mathcal{L})}(\Sigma^{-1}\tilde{C}_\bullet, \Sigma^m H_m(Q_\bullet)) \to Hom_{D^b_{\mathcal{L}}(\mathcal{T})}(\Sigma^{-1}\tilde{C}_\bullet, \Sigma^m H_m(Q_\bullet)) \cong Hom_{D^b_{\mathcal{L}}(\mathcal{T})}(\Sigma^{-1}\tilde{C}_\bullet, Q_\bullet)$$

is surjective. Thus there exists $\tilde{\alpha} : \Sigma^{-1}\tilde{C}_\bullet \to \Sigma^m H_m(Q_\bullet)$ with cone $\tilde{P}_\bullet$ such that the following diagram commutes

$$\begin{tikzcd}
\Sigma^{-1}C_\bullet \arrow[r, "\alpha"] \arrow[d, "\wr"] & Q_\bullet \arrow[r, "\beta"] \arrow[d, "\wr"] & P_\bullet \arrow[r, "\gamma"] & C_\bullet \arrow[d, "\wr"] \\
\iota(\Sigma^{-1}\tilde{C}_\bullet) \arrow[r, "\iota(\tilde{\alpha})"] & \iota(\Sigma^m H_m(Q_\bullet)) \arrow[r, "\iota(\tilde{\beta})"] & \iota(\tilde{P}_\bullet) \arrow[r, "\iota(\tilde{\gamma})"] & \iota(\tilde{C}_\bullet)
\end{tikzcd}$$

By the axiom TR3 of triangulated categories, $\iota(\tilde{P}_\bullet) \cong P_\bullet$. Hence (1) is proved.

We now give an idea of the proof of (2). Let $X_\bullet, Y_\bullet \in D^b(\mathcal{T}\cap\mathcal{L})$ and $f \in Hom_{D^b_{\mathcal{L}}(\mathcal{T})}(X_\bullet, Y_\bullet)$ and min$(X_\bullet \oplus Y_\bullet) = m$. We may assume, w.l.o.g., $min_c(X_\bullet) \geq m$ and $min_c(Y_\bullet) \geq m$. Applying Lemma 4.3, we get a morphism of triangles in $D^b_{\mathcal{L}}(\mathcal{T})$:

$$\begin{tikzcd}
\Sigma^{-1}C^X_\bullet \arrow[r, "\alpha^X"] \arrow[d, "\Sigma^{-1}\lambda"] & M^X_\bullet \arrow[r, "\beta^X"] \arrow[d, "\kappa"] & X_\bullet \arrow[r, "\gamma^X"] \arrow[d, "f"] & C^X_\bullet \arrow[d, "\lambda"] \\
\Sigma^{-1}C^Y_\bullet \arrow[r, "\alpha^Y"] & M^Y_\bullet \arrow[r, "\beta^Y"] & Y_\bullet \arrow[r, "\gamma^Y"] & C^Y_\bullet
\end{tikzcd}$$

where $M_\bullet^X, M_\bullet^Y$ are concentrated in degree m, the morphisms β^X, β^Y, κ are chain complex maps, $H_m(\beta^X), H_m(\beta^Y)$ are surjective and $\mathrm{Width}(C_\bullet^X \oplus C_\bullet^Y) < k$. Applying the induction hypothesis (2) to $C_\bullet^X$ and $C_\bullet^Y$, we get $\tilde{\lambda} \in Hom_{D^b(\mathcal{T}\cap\mathscr{L})}(C_\bullet^X, C_\bullet^Y)$ such that $\iota(\tilde{\lambda}) = \lambda$. We thus get a commutative diagram in $D^b(\mathcal{T}\cap\mathscr{L})$

$$\begin{tikzcd}
\Sigma^{-1}C_\bullet^X \arrow[r, "\alpha^X"] \arrow[d, "\Sigma^{-1}\tilde{\lambda}"] & M_\bullet^X \arrow[r, "\beta^X"] \arrow[d, "\kappa"] & X_\bullet \arrow[r, "\gamma^X"] \arrow[d, dashed, "g"] & C_\bullet^X \arrow[d, "\tilde{\lambda}"] \\
\Sigma^{-1}C_\bullet^Y \arrow[r, "\alpha^Y"] & M_\bullet^Y \arrow[r, "\beta^Y"] & Y_\bullet \arrow[r, "\gamma^Y"] & C_\bullet^Y
\end{tikzcd}$$

where g exists by axiom TR3 of triangulated categories. It still needs to be checked that $\iota(g) = f$. By linearity, we get a morphism of triangles in $D^b_{\mathscr{L}}(\mathcal{T})$ as above with the vertical arrow $f - \iota(g)$ from $X_\bullet$ to $Y_\bullet$ and other vertical arrows being 0. Using the weak kernel/cokernel properties, we get maps from $C_\bullet^X$ to $Y_\bullet$ and $X_\bullet$ to $M_\bullet^Y$. A very careful analysis of the Hom sets between these complexes and the induction hypothesis finally yields that $f = \iota(h)$ for some h, thus completing the proof. $\square$

Theorem 4.5. *Suppose $\mathscr{L}$ has the SR property. Then for a resolving subcategory $\mathcal{A}$, there is an equivalence of categories $D^b(\overline{\mathcal{A}}\cap\mathscr{L}) \rightsquigarrow D^b_{\mathscr{L}}(\mathcal{A})$. In particular, $D^b(\overline{\mathcal{P}}\cap\mathscr{L}) \simeq D^b_{\mathscr{L}}(\mathcal{P})$.*

Proof. Applying Theorem 4.4 with $\mathcal{T} = \overline{\mathcal{A}}$, we get that $D^b(\overline{\mathcal{A}}\cap\mathscr{L}) \simeq D^b_{\mathscr{L}}(\overline{\mathcal{A}})$. For a resolving subcategory $\mathcal{A}$, the natural functor $D^b_{\mathscr{L}}(\mathcal{A}) \rightsquigarrow D^b_{\mathscr{L}}(\overline{\mathcal{A}})$ is an equivalence using a standard total complex argument. Hence $D^b(\overline{\mathcal{A}}\cap\mathscr{L}) \simeq D^b_{\mathscr{L}}(\mathcal{A})$. $\square$

The following lemma gives a sufficient condition for a Serre subcategory to have the SR property in terms of ideals having efpd.

Lemma 4.6. *Let $\mathscr{L}$ be a Serre subcategory with the property that whenever $R/K' \in \mathscr{L}$, there exists $K \subseteq K'$ such that $R/K \in \mathscr{L}$ and K has efpd. Then $\mathscr{L}$ has the SR property.*

Proof. Let $X_\bullet \in Ch^b_{\mathscr{L}}(mod(R))$ and $J \subseteq R$ be an ideal such that $X_\bullet$ is not exact and $R/J \in \mathscr{L}$. Without loss of generality, assume $\min(X_\bullet) = 0$. Then there exists a bounded below chain complex $F_\bullet$ of free modules and a quasi-isomorphism $\pi : F_\bullet \to X_\bullet$ where $F_i = 0$ for all $i < 0$. It follows from [Wei94, Cor. 10.4.7] that

$$End_{K(R)}(F_\bullet) \cong End_{D(R)}(F_\bullet) \cong Hom_{D(R)}(F_\bullet, X_\bullet) \cong Hom_{K(R)}(F_\bullet, X_\bullet).$$

Since $X_\bullet$ is bounded, we get $Hom_{K(R)}(F_\bullet, X_\bullet)$ is finitely generated and hence, so is $End_{K(R)}(F_\bullet)$. Also, the map $(End_{K(R)}(F_\bullet))_p \to End_{K(R_p)}((F_\bullet)_p)$ is injective for all $p \in Spec(R)$, which follows from the above isomorphism and boundedness assumption on $X_\bullet$.

We claim that $Supp(End_{K(R)}(F_\bullet)) \subseteq \bigcup_{i=0}^{\infty} Supp(H_i(F_\bullet))$. Let $p \in Spec(R)$ such that $(End_{K(R)}(F_\bullet))_p \neq 0$. Then $End_{K(R_p)}((F_\bullet)_p) \neq 0$ and hence $(F_\bullet)_p$ is not homotopic to $O_\bullet$. Therefore $(F_\bullet)_p$ is not acyclic, since $F_\bullet$ is bounded below. So $p \in Supp(H_i(F_\bullet))$ for some i and hence the claim. Therefore we get

$$Supp(End_{K(R)}(F_\bullet)) \subseteq \bigcup_{i=0}^{\infty} Supp(H_i(F_\bullet)) = \bigcup_{i=0}^{\infty} Supp(H_i(X_\bullet)) \subseteq V_{\mathscr{L}}$$

where $V_{\mathscr{L}}$ is the specialization closed set corresponding to $\mathscr{L}$ as defined in 2.17(4).

Define $I' = Ann(End_{K(R)}(F_\bullet)) \subseteq R$. Then $V(I') = Supp(End_{K(R)}(F_\bullet)) \subseteq V_{\mathscr{L}}$. Since we assumed $R/J \in \mathscr{L}$, $V(J) = Supp(R/J) \subseteq V_{\mathscr{L}}$. Hence

$$V(I' \cap J) = V(I') \cup V(J) \subseteq V_{\mathscr{L}}.$$

Therefore, $R/(I' \cap J) \in \mathscr{L}$. By the assumption on $\mathscr{L}$, there exists $K \subseteq (I' \cap J)$ such that K has efpd and $R/K \in \mathscr{L}$. Let $f_1, \ldots, f_t$ generate K. Since $K \subseteq I' = Ann(End_{K(R)}(F_\bullet))$, $f_i Id_{F_\bullet}$ is null-homotopic for each i. A careful observation and reformulation in terms of the dual of the chain complex of [FH09, Proposition 23] gives a chain complex map

$$\phi : K_\bullet = Kos(f_1, \ldots, f_t) \otimes_R F_0 \to F_\bullet$$

such that $\min_c(K_\bullet) = 0$ and $\phi_0 : K_0 \to F_0$ is an isomorphism. Hence $H_0(\phi)$ is surjective.

Composing the above map ϕ with the map obtained from Theorem 3.5, there exist $u \in \mathbb{N}$ and a chain complex map $\psi : T(f_1^u, \ldots, f_t^u) \otimes_R F_0 \to F_\bullet$ such that ψ_0 is an isomorphism and hence $H_0(\psi)$ is surjective. Since K has efpd, there exists a filtration $\{K_n\}_{n\in\mathbb{N}}$ of K equivalent to the filtration $\{K^l\}_{l\in\mathbb{N}}$ and hence equivalent to the square-power filtration $\{K^{[l]}\}_{l\in\mathbb{N}}$ such that $pd(R/K_n) < \infty$ for all n. Then there exists $s \in \mathbb{N}$ such that $K_s \subseteq K^{[u]}$. This induces a chain complex map from the Tate resolution of R/K_s to the Tate resolution of $R/K^{[u]}$. Since $pd(R/K_s) < \infty$, the canonical truncation $T'_\bullet$ of its Tate resolution at that degree is homotopy equivalent to it via the natural inclusion and consists of projective modules. Define $T_\bullet = T'_\bullet \otimes F_0$, $I = K_s$ and α to be the chain complex map obtained by composition as follows :

$$T_\bullet \hookrightarrow T(R/K_s) \otimes F_0 \to T(f_1^u, \ldots, f_t^u) \otimes_R F_0 \xrightarrow{\psi} F_\bullet \quad .$$

It is now straightforward to see that $T_0 = F_0$ and the map at the zeroth degree is an isomorphism. Hence $H_0(\alpha)$ is surjective. Note that $\min_c(T_\bullet) = 0$ and $T_\bullet$ is a finite projective resolution of $R/K_s \otimes_R F_0$. Hence, $T_\bullet \in Ch^b_{\mathscr{L}}(\mathcal{P})$, and $\mathrm{Supph}(T_\bullet) = \{0\}$. Further, it follows that $H_0(T_\bullet) \cong T_0/IT_0$. Finally, $I = K_s \subseteq K^{[u]} \subseteq K \subseteq J$. Therefore, $(T_\bullet, \alpha, I)$ is a strong reducer of $(X_\bullet, J)$. □

Finally, as a straightforward consequence of Theorem 4.5 and Lemma 4.6, we obtain the main theorem of this section, which appears as Theorem 1.1.

Theorem 4.7. *Let R be a commutative Noetherian ring. Let $\mathcal{A} \subseteq mod(R)$ be a resolving subcategory. Then there is an equivalence of categories $D^b(\overline{\mathcal{A}} \cap \mathscr{L}) \rightsquigarrow D^b_{\mathscr{L}}(\mathcal{A})$ in the following cases :*

(1) R is a regular ring.
(2) $\mathscr{L}$ satisfies condition $(*)$ defined in [SS17, Definition 2.11].
(3) $\mathscr{L} = \mathscr{L}_{V(K)}$ where K has efpd.

In particular, $D^b(\overline{\mathcal{P}} \cap \mathscr{L}) \simeq D^b_{\mathscr{L}}(\mathcal{P})$ in these cases.

Proof. It is easy to check that in the mentioned cases, $\mathscr{L}$ satisfies the hypothesis of the Lemma 4.6 and thus has the SR property. Applying Theorem 4.5 completes the proof. □

5. Remarks on ideals of eventually finite projective dimension

5.1. **Some consequences of the derived equivalence.** We now study when an ideal I in a Noetherian ring has efpd. The next lemma exhibits a sequence of implications that give necessary conditions for an ideal to have eventually finite projective dimension.

Lemma 5.1. *Let R be a Noetherian ring, $I \neq R$ be an ideal in it and $\mathscr{L} = \mathscr{L}_{V(I)}$. Consider the following statements.*

(i) I is an ideal of eventually finite projective dimension.
(ii) There is a derived equivalence $D^b(\overline{\mathcal{P}} \cap \mathscr{L}) \simeq D^b_{\mathscr{L}}(\mathcal{P})$.
(iii) There exists a non-zero finitely generated module M with finite projective dimension and $Supp(M) = V(I)$.
(iv) For every minimal prime $\mathfrak{p}$ of I, $R_{\mathfrak{p}}$ is Cohen-Macaulay.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. $(i) \Rightarrow (ii)$: This is Theorem 4.7.
$(ii) \implies (iii)$: By the Hopkins-Neeman theorem,

$$V(I) = \bigcup_{X \in D^b_{\mathscr{L}}(\mathcal{P})} Supp(X) = \bigcup_{X \in D^b(\overline{\mathcal{P}} \cap \mathscr{L})} Supp(X) = \bigcup_{M \in \overline{\mathcal{P}} \cap \mathscr{L}} Supp(M).$$

Hence, for each $\mathfrak{p}_i \in Min(I)$, there exists $M_i \in \overline{\mathcal{P}} \cap \mathscr{L}$ such that $(M_i)_{\mathfrak{p}_i} \neq 0$. Thus $M = \oplus M_i$ is finitely generated, having finite projective dimension and $Supp(M) = V(I)$.
$(iii) \implies (iv)$: For any minimal prime $\mathfrak{p}$ of I, the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ has finite projective dimension and $Supp(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$. Hence by Corollary 2.9, $R_\mathfrak{p}$ is Cohen-Macaulay. ☐

Note that the implication $(i) \implies (iii)$ follows directly from the definition (e.g. by choosing $M = R/I_n$).

Question 5.2. Are any of the reverse implications true?

We suspect that $(iv) \not\Rightarrow (iii)$ but do not currently have a counter-example. We highlight below the special case where $(R, \mathfrak{m})$ is a local ring and I is an $\mathfrak{m}$-primary ideal wherein the statements above are equivalent. This is really a direct extension of [SS17, Theorem 1.1], which was really the inspiration for the previous lemma, and adds to the characterization of Cohen-Macaulay local rings in terms of the efpd property.

Corollary 5.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then the following are equivalent :*
- *(i) $\mathfrak{m}$ has efpd.*
- *(ii) $D^b(\overline{\mathcal{P}} \cap \mathscr{L}) \simeq D^b_{\mathscr{L}}(\mathcal{P})$ where $\mathscr{L} = \mathscr{L}_{V(\mathfrak{m})}$.*
- *(iii) There exists a non-zero finitely generated module M with finite length finite projective dimension.*
- *(iv) $(R, \mathfrak{m})$ is Cohen-Macaulay.*

Proof. By the previous lemma, we only need to prove $(iv) \Rightarrow (i)$ which follows by observing that when R is Cohen-Macaulay, then $\mathfrak{m}$ is a set-theoretic complete intersection ideal. ☐

5.2. **Known results and counter-examples related to ideals having efpd.** The definition of ideals of eventually finite projective dimension raises some natural questions regarding the eventual behaviour of the projective dimension of a filtration. The following consequence of [Bro79, Theorem 2(i)] is further suggestive that the limiting behaviour may be interesting.

Theorem 5.4. *Let R be a Noetherian ring and I be an ideal in R. Then $pd(R/I^n)$ is constant (possibly infinite) for all large $n \in \mathbb{N}$.*

Unfortunately, this property is special to the power filtration and is not shared by other equivalent filtrations such as the symbolic power filtrations, as demonstrated by the following recent result.

Theorem. [NT19, Theorem 6.3] *For any positive numerical function $\psi(t)$ which is periodic for $t \gg 0$, there exists a polynomial ring R and a homogeneous ideal $I \subset R$ such that $pd(R/I^{(n)}) = \psi(n) + c$ for some $c \geq 0$.*

Let R be the polynomial ring, and I the ideal obtained from the conclusion of the above theorem when ψ be a periodic function having period exactly 2. Clearly I has efpd, but considering the filtrations $\{J_n = I^{(n)}\}$, $\{J'_n = I^{(2n)}\}$ and $\{J''_n = I^{(2n+1)}\}$, all of which are equivalent to $\{I^n\}$, we see that $\lim_{n\to\infty} pd(R/J_n)$ does not exist, while $\lim_{n\to\infty} pd(R/J'_n)$ and $\lim_{n\to\infty} pd(R/J''_n)$ exist but converge to different values. These examples show that :

(1) I has efpd $\not\Rightarrow \lim_{n\to\infty} pd(R/J_n)$ exists for any filtration equivalent to $\{I^n\}$.

(2) I has efpd $\not\Leftarrow \lim_{n\to\infty} pd(R/J_n) = \lim_{n\to\infty} pd(R/J'_n)$ for filtrations $\{J_n\}$ and $\{J'_n\}$ equivalent to $\{I^n\}$ such that both limits exist.

The following recent result also shows that the power filtration may not yield information about whether an ideal has efpd.

Lemma. [KP21, Lemma 3.1] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. Let I be an $\mathfrak{m}$-primary and non-principal ideal, then $pd(R/I^n) = \infty$ for all $n \geq 1$.*

Since R is a Cohen-Macaulay local ring of dimension one, there exists an $\mathfrak{m}$-primary ideal J generated by an R-regular element. Therefore, $\{J^n\}_{n\gg 0}$ gives a filtration of I equivalent to the $\{I^n\}$-filtration, consisting of projective modules, and hence I has efpd.

To summarize, it appears that characterizing when an ideal has efpd through a filtration-independent numerical invariant based on their eventual behaviour is not possible.

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