SMALLNESS CONDITION IN THE WKB

Tarun K. Chandrayadula
IIT Madras

We want to express the smallness condition:

$$\left|\frac{1}{k_z^2(z)A(z)}\frac{d^2A(z)}{dz^2}\right| \ll 1 \tag{1}$$

in terms of $k_z(z)$. In doing so we want to see what are the angular frequencies (ω) and propagation angles ($\theta(z)$) for which the WKB approximation is valid. The discussion mostly follows George Frisk's book [1]. I have tried to explain all the intermediary steps, for the sake of students. Also I think he misses a factor of 2 in his Eq. 7.27 of the book.

This is not a complete derivation of the WKB criterion but rather a part of the derivation. For a derivation of the WKB criterion in the ocean acoustics context please refer to reference [1]

Given that

$$A(z) = \frac{D}{\sqrt{k_z(z)}}, \tag{2}$$

where D is a constant, we differentiate:

$$A(z) = Dk_z(z)^{-\frac{1}{2}}. \tag{3}$$

First derivative:

$$\frac{dA}{dz} = -\frac{D}{2}k_z(z)^{-\frac{3}{2}}k_z^{'}(z). \tag{4}$$

Second derivative:

$$\frac{d^2A}{dz^2} = -\frac{D}{2}\left(k_z(z)^{-\frac{3}{2}}k_z^{''}(z) - \frac{3}{2}k_z(z)^{-\frac{5}{2}}k_z^{'}(z)^2\right). \tag{5}$$

Now, compute $k_z^2(z)A(z)$:

$$k_z^2(z)A(z) = Dk^{\frac{3}{2}}(z). \tag{6}$$

Dividing,

$$\frac{1}{k_z^2(z)A(z)}\frac{d^2A}{dz^2} = -\frac{1}{2}k^{-3}(z)k_z^{''}(z) + \frac{3}{4}k^{-4}(z)k_z^{'}(z)^2. \tag{7}$$

Introduce $h(z) = \ln k_z(z)$, to write:

$$k_z^{''}(z) = k_z(z)h''(z) + k_z(z)h'(z)^2. \tag{8}$$

Rewriting our condition (Eq. 1):

$$\left|-\frac{1}{2}k^{-3}(z)\left(k_z(z)h''(z)+k_z(z)h'(z)^2\right)+\frac{3}{4}k^{-4}(z)\left(k_z(z)h'(z)\right)^2\right| \ll 1. \quad (9)$$

or,

$$\left|-\frac{1}{2}k^{-2}(z)h''(z)+\frac{1}{4}k^{-2}(z)h'(z)^2\right| \ll 1. \quad (10)$$

Thus, the smallness condition can be written as:

$$\left|\frac{1}{k_z^2(z)}\left(-\frac{1}{2}h''(z)+\frac{1}{4}h'(z)^2\right)\right| \ll 1. \quad (11)$$

or putting back the expression for $h(z) = ln(k_z(z)$ we get

$$\left|\frac{1}{k_z^2(z)}\left(2\frac{d^2}{dz^2}(ln(k_z(z))) - \left[\frac{d(ln(k_z(z))}{dz}\right]^2\right)\right| \ll 4. \quad (12)$$

Taking the first and second derivatives of $h(z)$ in Eq. 12,

$$\left|\frac{1}{k_z^2(z)}\left(2\frac{k_z''(z)}{k_z(z)} - 3\left(\frac{k_z'(z)}{k_z(z)}\right)^2\right)\right| \ll 4. \quad (13)$$

For locally linear such that $k_z''(z) \approx 0$ we have

$$\frac{1}{k_z(z)}\left|\frac{d}{dz}ln(k_z(z))\right| \ll \frac{2}{\sqrt{3}} \approx 1 \quad (14)$$

Now if we have $k_z(z) = k(z)cos(\theta(z))$, then we take the natural logarithm:

$$\ln k_z(z) = \ln k(z) + \ln\cos(\theta(z)). \quad (15)$$

Differentiating both sides of Eq. 15 with respect to z:

$$\frac{d}{dz}\ln k_z(z) = \frac{1}{k(z)}\frac{dk(z)}{dz} + \frac{1}{\cos(\theta(z))}\frac{d}{dz}\cos(\theta(z)). \quad (16)$$

Since

$$\frac{d}{dz}\cos(\theta(z)) = -\sin(\theta(z))\theta'(z), \quad (17)$$

we obtain:

$$\frac{d}{dz}\ln k_z(z) = \frac{k'(z)}{k(z)} - \tan(\theta(z))\theta'(z). \quad (18)$$

Now, dividing by $k_z(z)$:

$$\frac{1}{k_z(z)}\frac{d}{dz}\ln(k_z(z)) = \frac{1}{k(z)\cos(\theta(z))}\left(\frac{k'(z)}{k(z)} - \tan(\theta(z))\theta'(z)\right). \quad (19)$$

Expanding the term on the RHS within the braces as :

$$\frac{k'(z)}{k^2(z)\cos(\theta(z))} - \frac{\theta'(z)\sin(\theta(z))}{k(z)\cos^2(\theta(z))}, \tag{20}$$

the smallness condition

$$\left|\frac{1}{k_z(z)}\frac{d}{dz}\ln(k_z(z))\right| \ll 1 \tag{21}$$

becomes

$$\left|\frac{k'(z)}{k^2(z)\cos(\theta(z))} - \frac{\theta'(z)\sin(\theta(z))}{k(z)\cos^2(\theta(z))}\right| \ll 1. \tag{22}$$

If the rays are vertical (steep) such that $\theta(z) = 0$ we get

$$\left|\frac{k'(z)}{k^2(z)\cos(\theta(z))}\right| \ll 1. \tag{23}$$

We have $k(z) = \omega/c(z)$ for which

$$\frac{1}{\omega}\left|\frac{dc(z)}{dz}\right| \ll 1 \tag{24}$$

Eq. 24 above is the "smallness condition" that we set out to derive. For the WKB method to work we need the sound speed gradient to be small, and also the frequency to be high. Also WKB will not work for $\theta(z) = \pi/2$, which happens at turning points. The WKB is thus a no reflection theory.

References

[1] G. Frisk, *Ocean and Seabed Acoustics*. Englewood Cliffs, NJ: Prentice Hall, 1994.